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THE DECAY OF MAGNETO-TURBULENCE IN THE PRESENCE OF A MAGNETIC FIELD AND CORIOLIS FORCE*

BY

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Abstract. The final period of decay of magneto-turbulence in an external, homogeneous magnetic field is considered and it is shown that it develops pronounced axisymmetric properties, turbulence elements with finite wave numbers in the direction of the field being damped strongly under normal physical conditions. The turbulence consists of aperiodic motions as well as wave motions. An introduction of an angular velocity, inclined to the field, destroys the axisymmetry and modifies the damping effects and periodicity. The influence of the magnetic field on the damping is counteracted by the Coriolis force. A linear stationary theory on the action of the field gives results consistent with those of the theory of decay. From the results of both theories an explanation is given of the observed inhibition of turbulence in mercury by a magnetic field.

I. Introduction. From the large linear dimensions which one encounters in cosmical physics one may infer turbulence as basic for many applications in this field. In most applications the electrical conductivity is high enough for electrodynamic forces to play an essential role and for turbulence to be governed by the laws of magneto-hydrodynamics. However, it has been pointed out by Chandrasekhar[1] that even a small angular velocity of a medium of cosmical dimensions may sometimes be associated with a Coriolis force of the same importance as the electrodynamic force. In the general case a systematic magnetic field, created by constant sources, has also to be taken into account and turbulent fluctuations have to be superposed on the field.

Even if the influence of the boundaries is neglected, the general turbulent state is neither homogeneous, nor isotropic. The variation of mean velocity with position due to rotation destroys homogeneity also when the mean magnetic field is homogeneous. In this paper, however, we shall discuss turbulence in an incompressible liquid, where the centrifugal potential plays no essential role and regions are considered which are sufficiently small to justify the assumption of a homogeneous external field.

The problem of homogeneous, isotropic magneto-turbulence has been treated by Batchelor[2], Chandrasekhar[3] and Lundquist[4] among others. The second of these papers

*Received March 19, 1954.

[1]George Darwin lecture for 1953 (in press).

[2]Proc. Roy. Soc. A. 201, 405 (1950).

[3]Proc. Roy. Soc. A. **204**, 435 (1951); Ibid. **207**, 306 (1951).

[4]Arkiv f. fysik 5, 338 (1952).

gives a treatment in terms of invariant theory and the third presents a decay law for the spectral tensors for large wave numbers.

The purpose of this paper is mainly to discuss how the law of decay of homogeneous magneto-turbulence is modified by the introduction of an external magnetic field of the strength **B** and of the Coriolis force due to a constant angular velocity $\mathbf{\Omega}$ (MKSA-units are used in the following). Small amplitudes will be assumed and triple correlations will be neglected.

In Sec. III the law of decay in a homogeneous magnetic field is derived and it is shown that the motion becomes axisymmetric with respect to the direction of the field. In Sec. IV the analogous treatment in the presence of a constant angular velocity in a direction making an angle with the magnetic field is given. In Sec. V a brief discussion of stationary turbulence is given. The results are used in Sec. VI for an interpretation of the experimental results of Hartmann[5,6] and Lehnert[7,8] on turbulence in mercury.

II. The fundamental equations. The velocity field $\mathbf{v}$, in an incompressible liquid with constant electrical conductivity σ, kinematic viscosity ν, absolute permeability μ and density ρ, is supposed to be nonrelativistic and the liquid is assumed to rotate with a constant angular velocity $\mathbf{\Omega}$ in a homogeneous magnetic field $\mathbf{B} = \mu\mathbf{H}$, making an angle with $\mathbf{\Omega}$ and produced by external sources. The conductivity is assumed to be so good and the rate of change with time so slow that the displacement current can be neglected compared with the convection current $\mathbf{j}$, which is the source of the induced magneto-motive force $\mathbf{h}$. We start with the equations

$$\operatorname{curl}\mathbf{h} = \mathbf{j}, \qquad \operatorname{curl}\mathbf{E} = -\mu\frac{\partial\mathbf{h}}{\partial t}, \tag{1}$$

$$\operatorname{div}\mathbf{h} = \operatorname{div}\mathbf{v} = 0, \tag{2}$$

$$\mathbf{j} = \sigma[\mathbf{E} + \mu\mathbf{v}\times(\mathbf{H}+\mathbf{h})] \tag{3}$$

and

$$\rho\left[\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + 2\mathbf{\Omega}\times\mathbf{v} + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{x})\right] = \mu\mathbf{j}\times(\mathbf{H}+\mathbf{h}) + \nu\rho\nabla^2\mathbf{v} - \nabla(p + \rho\phi_g), \tag{4}$$

where $\mathbf{E}$ is the electric field, p the pressure, $\mathbf{x} = (x_1, x_2, x_3)$ the radius vector from an origin on the axis of rotation and ϕ_g the gravitation potential. The reduction of the system into two equations is similar to the treatment by Walén[9] and Lundquist[4] and will not be given here in detail. If the z-axis is chosen in the direction of the angular velocity $\mathbf{\Omega}$, the equations become

$$\frac{\partial\mathbf{V}}{\partial t} = (\mathbf{W}\cdot\nabla)\mathbf{v} + \lambda\nabla^2\mathbf{V} + \operatorname{curl}(\mathbf{v}\times\mathbf{V}) \tag{5}$$

and

$$\frac{\partial\mathbf{v}}{\partial t} = (\mathbf{W}\cdot\nabla)\mathbf{V} + \nu\nabla^2\mathbf{v} + 2\mathbf{v}\times\mathbf{\Omega} - \nabla\phi - \mathbf{V}\times\operatorname{curl}\mathbf{V} + \mathbf{v}\times\operatorname{curl}\mathbf{v}, \tag{6}$$

[5]Kgl. Danske Vidensk. Selskab Math.-fys. Medd. **15**, No. 6 (1937).
[6]J. Hartmann and F. Lazarus, Ibid. **15**, No. 7 (1937).
[7]Arkiv f. Fysik **5**, 69 (1952).
[8]Tellus **4**, 63 (1952).
[9]Arkiv f. mat., astr. o. fysik **30A**, No. 15 (1944).

where

$$\mathbf{V} = \mathbf{h}(\mu/\rho)^{1/2}, \qquad \mathbf{W} = \mathbf{H}(\mu/\rho)^{1/2}, \qquad \lambda = 1/(\mu\sigma), \tag{7}$$

$$\phi = p/\rho + \phi_g + \phi_c + \mathbf{W}\cdot\mathbf{V} + \rho v^2/2 \tag{8}$$

and

$$\phi_c = \tfrac{1}{2}\Omega^2(x_1^2 + x_2^2) \tag{9}$$

is the centrifugal potential.

For small amplitudes the terms of second order in Eqs. (5), (6) and (8) may be neglected. Further, letting

$$\mathbf{J} = \operatorname{curl} \mathbf{V}, \qquad \boldsymbol{\omega} = \operatorname{curl} \mathbf{v}, \tag{10}$$

and taking the curl of Eqs. (5) and (6) we obtain

$$\frac{\partial \mathbf{J}}{\partial t} = (\mathbf{W}\cdot\nabla)\boldsymbol{\omega} + \lambda\nabla^2\mathbf{J} \tag{11}$$

and

$$\frac{\partial \boldsymbol{\omega}}{\partial t} = (\mathbf{W}\cdot\nabla)\mathbf{J} + \nu\nabla^2\boldsymbol{\omega} + 2\boldsymbol{\omega}\times\boldsymbol{\Omega} + \nabla\psi, \tag{12}$$

where, from well-known vector operations,

$$\psi = 2\mathbf{v}\cdot\boldsymbol{\Omega}. \tag{13}$$

III. The law of decay in an external magnetic field. The decay of magneto-turbulence in an external magnetic field is of special interest in connexion with the experimental investigations on mercury. In an exact theory we must introduce triple correlations which represent the basic mechanism of turbulent interaction. However, it is known that they do not play a role in the final period and in this paper we shall restrict ourselves to this stage.

1. *Discussion of the gradient term.* We start with Eqs. (5) and (6) for a liquid, contained in a finite region and suppose that outside this region the conductivity and viscosity are zero while the permeability has the same value as in the inside. Since $\mathbf{v}$ and $\mathbf{V}$ are solenoidal vectors, we obtain on taking the divergence of Eq. (6) that

$$\nabla^2\phi = 0 \tag{14}$$

if all second order terms are neglected. Outside the liquid we have no terms representing the electromagnetic and the viscous forces and the result (14) continues to hold. Further, no surface currents are allowed to exist due to the finite value of the electrical conductivity. Thus, ϕ given by the expression (8) is continuous throughout the bounding surface. In the external region $\mathbf{V}$ tends to a dipole field at large distances from the liquid and ϕ tends to a constant value, representing the balance between pressure and gravitation force. Since ϕ is constant at infinity it is also constant on the boundary and in the interior of the liquid due to Green's theorm and

$$\nabla\phi = 0 \tag{15}$$

all over the liquid.

2. *The correlation tensors.* In terms of components Eqs. (5) and (6) can be written in the forms

$$\frac{\partial V_i}{\partial t} = W_k \frac{\partial v_i}{\partial x_k} + \lambda \nabla^2 V_i \tag{16}$$

and

$$\frac{\partial v_i}{\partial t} = W_k \frac{\partial V_i}{\partial x_k} + \nu \nabla^2 v_i , \tag{17}$$

where use has been made of Eq. (15). Summation over repeated indices is to be understood. Let

$$\mathbf{x}' = \mathbf{x} + \mathbf{r} \tag{18}$$

for a point at the distance $\mathbf{r}$ from $\mathbf{x}$. We shall distinguish the values of the various quantities at $\mathbf{x}'$ by a prime. Equations for the correlation tensors may now be formed in the usual manner; thus multiplying Eq. (16) by V_j' ; we get,

$$V_j' \frac{\partial V_i}{\partial t} = W_k V_j' \frac{\partial v_i}{\partial x_k} + \lambda V_j' \nabla^2 V_i \tag{19}$$

and adding to this equation that obtained from Eq. (19) by interchanging i and j and the primed and the unprimed quantities, we get

$$\frac{\partial}{\partial t} (V_i V_j') = W_k \left[\frac{\partial}{\partial x_k} (V_j' v_i) + \frac{\partial}{\partial x_k'} (V_i v_j') \right] + \lambda(\nabla^2 + \nabla'^2)(V_i V_j'). \tag{20}$$

In Eq. (20) (and similar equations in the sequel) one of the vectors $\mathbf{x}$ and $\mathbf{x}'$ is kept constant while the other is varied; then

$$\partial/\partial x_k' = \partial/\partial r_k = -\partial/\partial x_k \tag{21}$$

and Eq. (20) can be written in the form

$$\frac{\partial}{\partial t} (V_i V_j') = W_k \frac{\partial}{\partial r_k} (V_i v_j' - V_j' v_i) + 2\lambda \nabla^2 (V_i V_j'). \tag{22}$$

In the sequel all differential operators will refer to the variable $\mathbf{r}$.

Further equations are obtained in a similar manner by multiplying Eqs. (16) and (17) by v_j' . Introduce the tensors

$$M_{ij}(\mathbf{r}, t) = \tfrac{1}{2}\langle V_i(\mathbf{x}, t) V_j(\mathbf{x} + \mathbf{r}, t)\rangle, \tag{23}$$

representing the magnetic energy,

$$K_{ij}(\mathbf{r}, t) = \tfrac{1}{2}\langle v_i(\mathbf{x}, t) v_j(\mathbf{x} + \mathbf{r}, t)\rangle, \tag{24}$$

the kinetic energy, and

$$L_{ij}(\mathbf{r}, t) = \tfrac{1}{2}\langle V_i(\mathbf{x}, t) v_j(\mathbf{x} + \mathbf{r}, t)\rangle, \tag{25}$$

the interaction energy respectively. In forming these tensors the mean values are with respect to time. In homogeneous turbulence these tensors should be invariant to arbitrary displacements; this invariance leads to the following geometrical properties:

$$M_{ij}(\mathbf{r}) = \tfrac{1}{2}\langle V_i(\mathbf{x}' - \mathbf{r}) V_j(\mathbf{x}')\rangle = M_{ji}(-\mathbf{r}), \tag{26}$$

$$K_{ij}(\mathbf{r}) = K_{ji}(-\mathbf{r}) \tag{27}$$

and

$$L_{ji}(-\mathbf{r}) = \tfrac{1}{2}\langle V_j(\mathbf{x})v_i(\mathbf{x} - \mathbf{r})\rangle = \tfrac{1}{2}\langle V_j(\mathbf{x} + \mathbf{r})v_i(\mathbf{x})\rangle = L_{ji}(\mathbf{x}', \mathbf{x}). \tag{28}$$

Further, M_{ij} and K_{ij} are symmetric, whereas L_{ij} is skewsymmetric. $M_{ii}(0)$ and $K_{ii}(0)$ are the mean magnetic and kinetic energies at the point $\mathbf{x}$ and time t. The physical significance of L_{ij} is easily seen from the electrodynamic force in Eq. (17). The decrease in magnetic energy per unit mass and time due to the work of the electrodynamic force is

$$E_{ii}^{(MH)} = W_k \frac{\partial V_i}{\partial x_k} v_i\ , \tag{29}$$

which is a positive quantity if magnetic energy is converted into kinetic energy. From Eqs. (21) and (28) we get the expression

$$E_{ii}^{(MH)} = \lim_{r\to 0} \frac{1}{2} W_k\left(v_i' \frac{\partial V_i}{\partial x_k} + v_i \frac{\partial V_i'}{\partial x_k'}\right) = \lim_{r\to 0}\left\{-W_k \frac{\partial}{\partial r_k}\left[L_{ii}(\mathbf{r}) - L_{ii}(-\mathbf{r})\right]\right\}. \tag{30}$$

Returning to Eq. (22) and taking the mean value we get

$$\frac{\partial}{\partial t} M_{ij} = W_k \frac{\partial}{\partial r_k}\left[L_{ij}(\mathbf{r}) - L_{ji}(-\mathbf{r})\right] + 2\lambda\nabla^2 M_{ij}\ . \tag{31}$$

Similarly we obtain

$$\frac{\partial}{\partial t} K_{ij} = -W_k \frac{\partial}{\partial r_k}\left[L_{ij}(\mathbf{r}) - L_{ji}(-\mathbf{r})\right] + 2\nu\nabla^2 K_{ij} \tag{32}$$

and

$$\frac{\partial}{\partial t} L_{ij}(\mathbf{r}) = W_k \frac{\partial}{\partial r_k}(M_{ij} - K_{ij}) + (\lambda + \nu)\nabla^2 L_{ij}(\mathbf{r}). \tag{33}$$

Interchanging $\mathbf{r}$ and $-\mathbf{r}$ and i and j in Eq. (33) we obtain

$$\frac{\partial}{\partial t} L_{ji}(-\mathbf{r}) = -W_k \frac{\partial}{\partial r_k}(M_{ij} - K_{ij}) + (\lambda + \nu)\nabla^2 L_{ji}(-\mathbf{r}). \tag{34}$$

Now, assume that a spectral tensor, $\Lambda_{ij}(\boldsymbol{\kappa})$, exists such that

$$\Lambda_{ij}(\boldsymbol{\kappa}) = (8\pi^3)^{-1} \iiint M_{ij}(\mathbf{r}) \exp(-i\boldsymbol{\kappa}\cdot\mathbf{r})\, dr_1\, dr_2\, dr_3 \tag{35}$$

and

$$M_{ij}(\mathbf{r}) = \iiint \Lambda_{ij}(\boldsymbol{\kappa}) \exp(i\boldsymbol{\kappa}\cdot\mathbf{r})\, d\kappa_1\, d\kappa_2\, d\kappa_3\ , \tag{36}$$

where $\boldsymbol{\kappa}$ is the wave number. A spectral kinetic energy tensor, $\boldsymbol{\Omega}_{ij}(\boldsymbol{\kappa})$, and a spectral interaction tensor, $\Upsilon_{ij}(\boldsymbol{\kappa})$, can be similarly defined in terms of $K_{ij}(\mathbf{r})$ and $L_{ij}(\mathbf{r})$ respectively.

Since differentiation with respect to r_k in ordinary space corresponds to multiplication with a factor $i\kappa_k$ in the wave-number space we have from Eqs. (31), (32), (33) and (34):

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + 2\lambda\kappa^2\right)\Lambda_{ij} - i\kappa_k W_k[\Upsilon_{ij}(\boldsymbol{\kappa}) - \Upsilon_{ji}(-\boldsymbol{\kappa})] &= 0,\\
\left(\frac{\partial}{\partial t} + 2\nu\kappa^2\right)\Omega_{ij} + i\kappa_k W_k[\Upsilon_{ij}(\boldsymbol{\kappa}) - \Upsilon_{ji}(-\boldsymbol{\kappa})] &= 0,\\
\left[\frac{\partial}{\partial t} + (\lambda + \nu)\kappa^2\right][\Upsilon_{ij}(\boldsymbol{\kappa}) - \Upsilon_{ji}(-\boldsymbol{\kappa})] - 2i\kappa_k W_k(\Lambda_{ij} - \Omega_{ij}) &= 0.
\end{aligned} \tag{37}$$

These equations are also valid for arbitrary amplitudes if sufficiently high wave numbers are considered.

3. *The law of decay.* The system of Eqs. (37) will be satisfied by solutions of the form exp (mt) if the determinant of the system

$$\begin{vmatrix} m+2a & 0 & -iF \\ 0 & m+2b & iF \\ -2iF & 2iF & m+a+b \end{vmatrix}, \tag{38}$$

where

$$a = \lambda\kappa^2; \qquad b = \nu\kappa^2; \qquad F = \kappa_k W_k, \tag{39}$$

vanishes. Hence

$$(m+a+b)[m^2 + 2(a+b)m + 4ab + 4F^2] = 0. \tag{40}$$

The roots of this equation are

$$m_{1,2} = -(a+b) \pm [(a-b)^2 - 4F^2]^{1/2}, \qquad m_3 = -(a+b) \tag{41}$$

and the solutions have the form

$$\begin{vmatrix} \Lambda_{ij}(\boldsymbol{\kappa}, t) \\ \Omega_{ij}(\boldsymbol{\kappa}, t) \\ \Pi_{ij}(\boldsymbol{\kappa}, t) \end{vmatrix} = \begin{vmatrix} \Lambda_{ij}^{(1)} & \Lambda_{ij}^{(2)} & \Lambda_{ij}^{(3)} \\ \Omega_{ij}^{(1)} & \Omega_{ij}^{(2)} & \Omega_{ij}^{(3)} \\ \Pi_{ij}^{(1)} & \Pi_{ij}^{(2)} & \Pi_{ij}^{(3)} \end{vmatrix} \cdot \begin{vmatrix} \exp(m_1 t) \\ \exp(m_2 t) \\ \exp(m_3 t) \end{vmatrix}, \tag{42}$$

where

$$\Pi_{ij}(\boldsymbol{\kappa}, t) = \Upsilon_{ij}(\boldsymbol{\kappa}, t) - \Upsilon_{ji}(-\boldsymbol{\kappa}, t) \tag{43}$$

has been introduced. When m has been chosen as a root of Eq. (40) the solution of the linear system (37) can be written in the form

$$\begin{vmatrix} \Lambda_{ij}(\boldsymbol{\kappa}, t) \\ \Pi_{ij}(\boldsymbol{\kappa}, t) \end{vmatrix}$$

$$= \begin{vmatrix} [1-(1-\zeta^2)^{1/2}]/[1+(1-\zeta^2)^{1/2}] & [1+(1-\zeta^2)^{1/2}]/[1-(1-\zeta^2)^{1/2}] & 1 \\ -2i[1-(1-\zeta^2)^{1/2}]/\zeta & -2i[1+(1-\zeta^2)^{1/2}]/\zeta & -2i/\zeta \end{vmatrix}$$

$$\cdot \begin{vmatrix} \Omega_{ij}^{(1)} \exp(m_1 t) \\ \Omega_{ij}^{(2)} \exp(m_2 t) \\ \Omega_{ij}^{(3)} \exp(m_3 t) \end{vmatrix}, \tag{44}$$

where we have introduced the parameter

$$\zeta = 2F/(a-b) = 2\kappa_k W_k/[\kappa^2(\lambda - \nu)] \tag{45}$$

and have written

$$m_{1,2} + 2a = a - b \pm (a - b)(1 - \zeta^2)^{1/2}, \qquad m_3 + 2a = a - b \tag{46}$$

and

$$m_{1,2} + 2b = -(a - b) \pm (a - b)(1 - \zeta^2)^{1/2}, \qquad m_3 + 2b = -(a - b). \tag{47}$$

For small values of ζ the form (44) reduces to

$$\begin{vmatrix} \Lambda_{ij}(\kappa, t) \\ \Pi_{ij}(\kappa, t) \end{vmatrix} = \begin{vmatrix} \zeta^2/4 & 4/\zeta^2 & 1 \\ -i\zeta & -4i/\zeta & -2i/\zeta \end{vmatrix} \cdot \begin{vmatrix} \Omega_{ij}^{(1)} \exp (m_1 t) \\ \Omega_{ij}^{(2)} \exp (m_2 t) \\ \Omega_{ij}^{(3)} \exp (m_3 t) \end{vmatrix}. \tag{48}$$

All quantities have to be finite at $\zeta = 0$, and therefore, when ζ tends to zero,

$$\Lambda_{ij}^{(1)} = \mathcal{O}(\zeta^2) \rightarrow 0, \qquad \Lambda_{ij}^{(2)} = \Lambda_{ij}^{(0)} + \mathcal{O}(\zeta), \qquad \Lambda_{ij}^{(3)} = \mathcal{O}(\zeta) \rightarrow 0, \tag{49}$$

$$\Omega_{ij}^{(1)} = \Omega_{ij}^{(0)} + \mathcal{O}(\zeta), \qquad \Omega_{ij}^{(2)} = \mathcal{O}(\zeta^2) \rightarrow 0, \qquad \Omega_{ij}^{(3)} = \mathcal{O}(\zeta) \rightarrow 0, \tag{50}$$

$$\Pi_{ij}^{(1)} = \mathcal{O}(\zeta) \rightarrow 0, \qquad \Pi_{ij}^{(2)} = \mathcal{O}(\zeta) \rightarrow 0, \qquad \Pi_{ij}^{(3)} = \Pi_{ij}^{(0)} + \mathcal{O}(\zeta), \tag{51}$$

where $\Lambda_{ij}^{(0)}$, $\Omega_{ij}^{(0)}$ and $\Pi_{ij}^{(0)}$ are independent of ζ and $\mathcal{O}(\zeta^n)$ are terms at least of order n. For $\zeta = 0$ we get

$$\Lambda_{ij}(\kappa, t) = \Lambda_{ij}^{(0)}(\kappa) \exp (-2\lambda\kappa^2 t), \tag{52}$$

$$\Omega_{ij}(\kappa, t) = \Omega_{ij}^{(0)}(\kappa) \exp (-2\nu\kappa^2 t) \tag{53}$$

and

$$\Pi_{ij}(\kappa, t) = \Pi_{ij}^{(0)}(\kappa) \exp [-(\lambda + \nu)\kappa^2 t]. \tag{54}$$

These solutions represent the final period of decay of isotropic turbulence and are analogous to the case studied by Lundquist where the magneto-hydrodynamic interaction, expressed by triple correlations, was neglected and the magnetic and kinetic turbulence fields were found to decay independently of each other. The introduction of an external magnetic field, however, changes the situation even in first order and the decay is mainly governed by a coupling of the form (29).

The solutions (41) define a decay time, τ_k, given by

$$1/\tau_{1,2} = \kappa^2(\lambda + \nu) \mp \kappa^2(\lambda - \nu)(1 - \zeta^2)^{1/2}; \qquad 1/\tau_3 = \kappa^2(\lambda + \nu). \tag{55}$$

Thus, for $\zeta^2 < 1$ we have three non-periodic solutions, all with different decay times; for $\zeta^2 = 1$ all the spectral tensors decay with the same time constant

$$\tau_c = \tau_3 = 1/[\kappa^2(\lambda + \nu)] \tag{56}$$

and finally for values of $\zeta^2 > 1$ two periodic solutions are obtained, both with the same real damping, $1/\tau_3$.

4. *Physical interpretation of the law of decay.* The physical meaning underlying the results (42), (44) and (55) can be understood from a consideration of a plane state of motion in a liquid between two infinitely conducting planes at a distance L, as shown

by Fig. 1. Introduce a homogeneous magnetic field $\mathbf{B}_0$ in the z-direction, perpendicular to the planes. In a plane state of motion and for small amplitudes

$$\partial/\partial x = \partial/\partial y = 0; \qquad \mathbf{v} = (0, v, 0); \qquad \mathbf{V} = (0, V, 0) \tag{57}$$

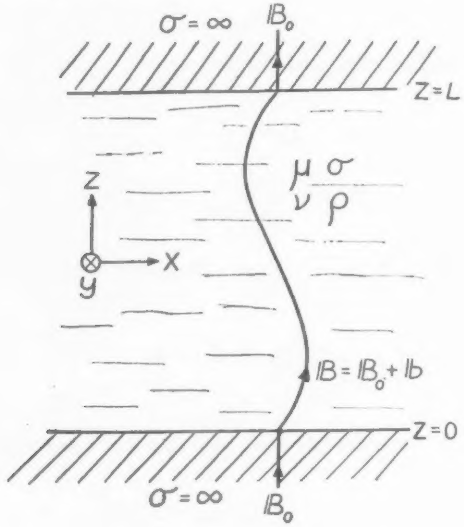

FIG. 1. Plane state of motion of an electrically conducting, viscous liquid between two infinitely conducting planes in a homogeneous, perpendicular magnetic field, $\mathbf{B}_0$.

and Eqs. (5) and (6) reduce to

$$\frac{\partial V}{\partial t} = W \frac{\partial v}{\partial z} + \lambda \frac{\partial^2 V}{\partial z^2} \tag{58}$$

and

$$\frac{\partial v}{\partial t} = W \frac{\partial V}{\partial z} + \nu \frac{\partial^2 v}{\partial z^2}. \tag{59}$$

On eliminating V we obtain the following equation:

$$\left[\frac{\partial^2}{\partial t^2} - W^2 \frac{\partial^2}{\partial z^2} - (\lambda + \nu) \frac{\partial^3}{\partial t\, \partial z^2} + \lambda\nu \frac{\partial^4}{\partial z^4}\right] v = 0. \tag{60}$$

Separating the variables in the form

$$v = Z(z)T(t) \tag{61}$$

we get

$$T''/T - W^2 Z''/Z - (\lambda + \nu)(Z''/Z)(T'/T) + \lambda\nu Z^{\mathrm{IV}}/Z = 0. \tag{62}$$

Since the magnetic field lines and the liquid are attached to the infinitely conducting walls we may assume that

$$Z = \sin\,[\pi k z/L] = \sin\,(\kappa z) \qquad (k = 1, 2, \cdots \kappa = \pi k/L) \tag{63}$$

in which case T admits a solution of the form

$$T = \exp\left(\tfrac{1}{2}mt\right), \tag{64}$$

provided

$$m^2 + 4\kappa^2W^2 + 2(\lambda + \nu)\kappa^2 m + 4\lambda\nu\kappa^4 = 0. \tag{65}$$

The roots of this equation are

$$m_{1,2} = -(\lambda + \nu)\kappa^2 \pm (\lambda - \nu)\kappa^2(1 - \zeta^2)^{1/2}, \tag{66}$$

which are the two first values given by Eq. (41). This corresponds to a motion of the magnetic lines of force, regarded as elastic strings with a tension given by the magnetic field strength and a damping due to the Joule heat and the viscous losses. For $\zeta^2 \leq 1$ the equivalent strings move aperiodically and for $\zeta^2 > 1$ damped waves travel along the strongs. Correlations may be formed by products of the solutions of Eq. (60), giving time factors of the form

$$T^{(I)} \cdot T^{(II)} = \exp\left[\tfrac{1}{2}(m_{1,2} + m_{1,2})t\right] \tag{67}$$

for a given value of κ. These factors are consistent with those given by Eqs. (41) and (42).

We shall now return to the discussion of the results (42). Whatever distribution we may start with at $t = 0$ these results will always tend to a solution, which for the kinetic tensor reduces to

$$\Omega_{ij}(\boldsymbol{\kappa}, t) = \Omega_{ij}^{(k)}(\boldsymbol{\kappa}) \exp\left(-t/\tau_k\right), \tag{68}$$

where $1/\tau_k$ is the value of the one of the decay factors (55) having the smallest real part. The solution (68) will differ for turbulence elements due to their size, the properties of the liquid and the strength of the external field.

For small values of ζ and for large ratios λ/ν ($\lambda/\nu \approx 10^5$ for experiments with mercury) the decay factors (55) become

$$1/\tau_1 \approx 2\nu\kappa^2 + \tfrac{1}{2}\kappa^2(\lambda - \nu)\zeta^2 \tag{69}$$

and

$$1/\tau_2 \approx 2\lambda\kappa^2 - \tfrac{1}{2}\kappa^2(\lambda - \nu)\zeta^2, \tag{70}$$

whereas $1/\tau_3$ remains unaltered; further, $1/\tau_1$ is much smaller than $1/\tau_2$ and $1/\tau_3$, if ζ is sufficiently small. Now, Eq. (50) shows that the corresponding factor, $\Omega_{ij}^{(1)}$, has a term of zero order in ζ and consequently $\Omega_{ij}(\boldsymbol{\kappa}, t)$ will be represented by the asymptotic law

$$\Omega_{ij}(\boldsymbol{\kappa}, t) \approx \Omega_{ij}^{(1)}(\boldsymbol{\kappa}) \exp\left\{-2[\nu\kappa^2 + \kappa_3^2W^2/\kappa^2(\lambda - \nu)]t\right\} \tag{71}$$

during the largest part of the final period of decay, if the x_3-axis is chosen in the direction of the field and Eq. (45) is used. Analogous discussions may be carried out when $\lambda < \nu$. The results of this section may be summed up in the following statements:

During the turbulent decay of a liquid with $\lambda \gg \nu$ all periodic turbulence elements as well as the aperiodic ones with small extensions in the direction of the field (large values of κ_3) *are damped out relatively rapidly. The asymptotic state of decay is two-dimensional* with respect to the direction of the field. Only vortices with $\kappa_3 = 0$ are left in the final state;

the damping is by viscosity only and since the electric field becomes independent of x_3 no induced currents flow.

The main kinetic decay factor, $1/\tau_1$, for mercury, given by Eq. (71) is shown in Fig. 2, where we have put $\kappa_1 = \kappa_2$. The curves clearly show action of the field in suppressing elements with finite wave numbers in the x_3-direction.

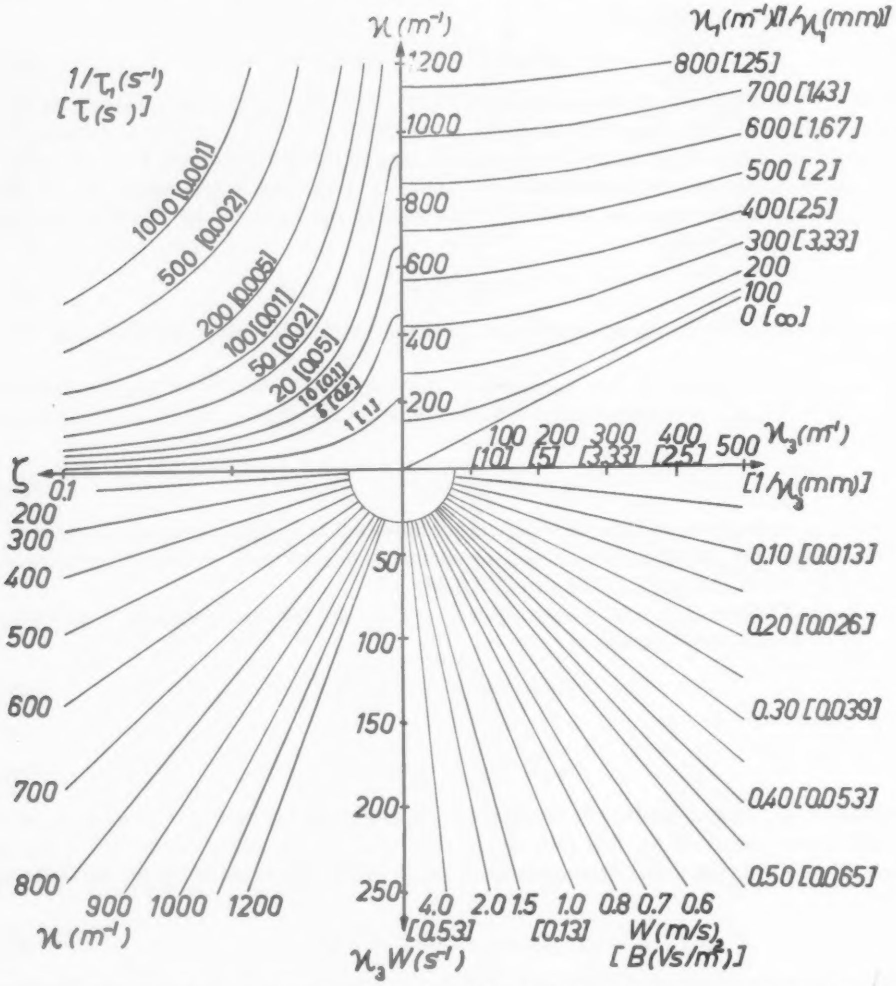

FIG. 2. The factor of decay, $1/\tau_1$, for the final period of turbulence in mercury as a function of an external magnetic field [$W = B_0(\mu\rho)^{-\frac{1}{2}}$ being the corresponding wave velocity] and the wave numbers $\kappa_1 = \kappa_2$ and κ_3. The values $\lambda = 0.763\ m^2/s$ and $\nu = 0.116 \times 10^{-6}\ m^2/s$ at 18°C have been used.

IV. The decay in a magnetic field in presence of the Coriolis force. It was pointed out at the beginning of this paper that the influence of the Coriolis force may sometimes be comparable to the electromagnetic force, both leading to first order effects which may be significant for small as well as for large turbulent intensities. The effect of these

forces, when they act separately on a liquid, shows many similarities. The stationary flow of a rotating liquid becomes two-dimensional at high angular velocities[10] and the same situation is true in a conducting liquid in a strong, homogeneous magnetic field.[7,8] Similarly both have inhibiting effects on the onset of convection.[11,12] But if both forces are simultaneously present a complicated situation arises, which is not the same as the superposition of the separate effects. This is evident for example[11] from Chandrasekhar's investigations of the thermal instability of a fluid layer heated below under the joint effects of a Coriolis acceleration and a magnetic field.

1. *The correlation tensors.* We shall now extend the theory of the decay of turbulence described in the preceeding sections to allow for the effect of a stationary angular velocity in the x_3-direction: thus

$$\mathbf{\Omega} = (0, 0, \Omega) = \tfrac{1}{2} \operatorname{curl} \mathbf{U}; \qquad \mathbf{U} = \mathbf{\Omega} \times \mathbf{x}, \tag{72}$$

where $\mathbf{U}$ is the velocity of rotation of the liquid and $\mathbf{x}$ is the radius vector from the origin on the axis of rotation. We choose the coordinate system with the magnetic field vector in the x_2, x_3-plane so that

$$\mathbf{W} = (0, W_2, W_3). \tag{73}$$

As we shall see it is most convenient to discuss this problem in terms of the current density and the vorticity. We now start with Eqs. (11) and (12), which written in terms of components have the forms

$$\frac{\partial J_i}{\partial t} = W_k \frac{\partial \omega_i}{\partial x_k} + \lambda \nabla^2 J_i \tag{74}$$

and

$$\frac{\partial \omega_i}{\partial t} = -W_k \frac{\partial J_i}{\partial t} + \nu \nabla^2 \omega_i + 2\epsilon_{ilm}\omega_l \Omega_m + \frac{\partial \psi}{\partial x_i}, \tag{75}$$

where ϵ_{ilm} is the usual alternating symbol. The equations governing the various correlations can be obtained in the same manner as in Sec. III; the only difference is that now we shall have terms containing $\mathbf{\Omega}$ added to the right hand sides of the various equations; thus

$$\frac{\partial}{\partial t}(\omega_i \omega_j') = \cdots + 2\omega_j' \epsilon_{ilm} \omega_l \Omega_m + 2\omega_i \epsilon_{jlm} \omega_l' \Omega_m + \omega_j' \frac{\partial \psi}{\partial x_i} + \omega_i \frac{\partial \psi'}{\partial x_j'}, \tag{76}$$

$$\frac{\partial}{\partial t}(J_i \omega_j') = \cdots + 2 J_i \epsilon_{jlm} \omega_l' \Omega_m + J_i \frac{\partial \psi'}{\partial x_j'}. \tag{77}$$

Introduce the tensors

$$R_{ij}(\mathbf{x}, \mathbf{x}') = \langle J_i(\mathbf{x}) J_j(\mathbf{x} + \mathbf{r}) \rangle \equiv R_{ji}(\mathbf{x}', \mathbf{x}), \tag{78}$$

$$T_{ij}(\mathbf{x}, \mathbf{x}') = \langle \omega_i(\mathbf{x}) \omega_j(\mathbf{x} + \mathbf{r}) \rangle \equiv T_{ji}(\mathbf{x}', \mathbf{x}), \tag{79}$$

$$S_{ij}(\mathbf{x}, \mathbf{x}') = \langle J_i(\mathbf{x}) \omega_j(\mathbf{x} + \mathbf{r}) \rangle, \tag{80}$$

$$P_{ij}(\mathbf{x}, \mathbf{x}') = \left\langle \frac{\partial}{\partial r_i} [J_j(\mathbf{x} + \mathbf{r}) \psi(\mathbf{x})] \right\rangle - \left\langle \frac{\partial}{\partial r_j} [J_i(\mathbf{x}) \psi(\mathbf{x} + \mathbf{r})] \right\rangle \tag{81}$$

[10] G. I. Taylor, Proc. Roy. Soc. **A100**, 114 (1921).
[11] S. Chandrasekhar, Phil. Mag. **43**, 501 (1952).
[12] S. Chandrasekhar Proc. Roy. Soc. **A217**, 306 (1953).

and

$$Q_{ij}^*(\mathbf{x}, \mathbf{x}') = \left\langle \frac{\partial}{\partial r_i} [\omega_i(\mathbf{x} + \mathbf{r})\psi(\mathbf{x})] \right\rangle; \qquad Q_{ij}(\mathbf{x}, \mathbf{x}') = Q_{ij}^*(\mathbf{x}, \mathbf{x}') - Q_{ji}^*(\mathbf{x}', \mathbf{x}), \tag{82}$$

where relations (18) and (21) have been used. These notations will be used for the present. They are valid also in the inhomogeneous case. The condition for homogeneity will be applied later. From Eqs. (76) and (77) we now find the equations governing the correlations:

$$\frac{\partial}{\partial t} R_{ij}(\mathbf{x}, \mathbf{x}') = W_k \frac{\partial}{\partial r_k} [S_{ij}(\mathbf{x}, \mathbf{x}') - S_{ji}(\mathbf{x}', \mathbf{x})] + 2\lambda \nabla^2 R_{ij}(\mathbf{x}, \mathbf{x}'), \tag{83}$$

$$\begin{aligned} \frac{\partial}{\partial t} T_{ij}(\mathbf{x}, \mathbf{x}') = &-W_k \frac{\partial}{\partial r_k} [S_{ij}(\mathbf{x}, \mathbf{x}') - S_{ji}(\mathbf{x}', \mathbf{x})] + 2\nu \nabla^2 T_{ij}(\mathbf{x}, \mathbf{x}') \\ &+ 2\Omega_m[\epsilon_{ilm} R_{lj}(\mathbf{x}, \mathbf{x}') + \epsilon_{jlm} R_{il}(\mathbf{x}, \mathbf{x}')] - P_{ij}(\mathbf{x}, \mathbf{x}') \end{aligned} \tag{84}$$

and

$$\begin{aligned} \frac{\partial}{\partial t} S_{ij}(\mathbf{x}, \mathbf{x}') = W_k \frac{\partial}{\partial r_k} [R_{ij}(\mathbf{x}, \mathbf{x}') - T_{ij}(\mathbf{x}, \mathbf{x}')] &+ (\lambda + \nu) \nabla^2 S_{ij}(\mathbf{x}, \mathbf{x}') \\ &+ 2\Omega_m \epsilon_{jlm} S_{il}(\mathbf{x}, \mathbf{x}') + Q_{ji}^*(\mathbf{x}', \mathbf{x}). \end{aligned} \tag{85}$$

Interchanging i and j and $\mathbf{x}$ and $\mathbf{x}'$ in Eq. (85) and subtracting the resulting equation from Eq. (85), we obtain

$$\begin{aligned} &\frac{\partial}{\partial t} [S_{ij}(\mathbf{x}, \mathbf{x}') - S_{ji}(\mathbf{x}', \mathbf{x})] \\ &= 2W_k \frac{\partial}{\partial r_k} [R_{ij}(\mathbf{x}, \mathbf{x}') - T_{ij}(\mathbf{x}, \mathbf{x}')] + (\lambda + \nu) \nabla^2 [S_{ij}(\mathbf{x}, \mathbf{x}') - S_{ji}(\mathbf{x}', \mathbf{x})] \\ &\qquad + 2\Omega_m[\epsilon_{jlm} S_{il}(\mathbf{x}, \mathbf{x}') - \epsilon_{ilm} S_{jl}(\mathbf{x}, \mathbf{x}')] - Q_{ij}(\mathbf{x}, \mathbf{x}'). \end{aligned} \tag{86}$$

2. *The spectral tensors and the law of decay.* Before the equations governing the correlations are transformed into a spectral representation we shall consider the properties of the tensors (81) and (82). It is easily seen that

$$P_{ii} = Q_{ii} = 0, \tag{87}$$

since $\mathbf{J}$ and $\boldsymbol{\omega}$ are solenoidal. If homogeneity is assumed, the tensor $Q_{ij}^*(\mathbf{x}, \mathbf{x}')$ will have the property

$$\begin{aligned} Q_{ij}^*(\mathbf{x}, \mathbf{x}') &= \left\langle \frac{\partial}{\partial r_i} [\omega_i(\mathbf{x} + \mathbf{r})\psi(\mathbf{x})] \right\rangle = \left\langle \frac{\partial}{\partial r_i} [\omega_i(\mathbf{x})\psi(\mathbf{x} - \mathbf{r})] \right\rangle \\ &= -\left\langle \frac{\partial}{\partial r_i} [\omega_i(\mathbf{x})\psi(\mathbf{x} + \mathbf{r})] \right\rangle = -Q_{ij}^*(\mathbf{x}', \mathbf{x}). \end{aligned} \tag{88}$$

From Eqs. (81) and (82) we get the result

$$Q_{ij}(\mathbf{x}, \mathbf{x}') - Q_{ji}(\mathbf{x}, \mathbf{x}') = P_{ij}(\mathbf{x}, \mathbf{x}') - P_{ji}(\mathbf{x}, \mathbf{x}') = 0. \tag{89}$$

Now introduce the Fourier transforms $\Phi_{ij}(\kappa)$, $\Psi_{ij}(\kappa)$, $\Gamma_{ij}(\kappa)$, $\chi_{ij}(\kappa)$, $\vartheta^*_{ij}(\kappa)$ and $\vartheta_{ij}(\kappa)$ of the tensors $R_{ij}(\mathbf{r})$, $T_{ij}(\mathbf{r})$, $S_{ij}(\mathbf{r})$, $P_{ij}(\mathbf{r})$, $Q^*_{ij}(\mathbf{r})$ and $Q_{ij}(\mathbf{r})$ respectively; thus

$$\Phi_{ij}(\kappa) = (8\pi^3)^{-1} \iiint R_{ij}(\mathbf{r}) \exp(-i\kappa\cdot\mathbf{r})\, dr_1\, dr_2\, dr_3\,, \qquad \text{etc.} \tag{90}$$

Also we shall write

$$\Gamma'_{ij} \equiv \Gamma_{ij}(-\kappa); \qquad \vartheta^{*\prime}_{ij} \equiv \vartheta^*_{ij}(-\kappa). \tag{91}$$

By applying the Fourier transforms to Eqs. (83), (84), (85) and (86) we obtain the following equations for the spectral tensors:

$$\left(\frac{\partial}{\partial t} + 2a\right)\Phi_{ij} - iF(\Gamma_{ij} - \Gamma'_{ji}) = 0, \tag{92}$$

$$\left(\frac{\partial}{\partial t} + 2b\right)\Psi_{ij} + iF(\Gamma_{ij} - \Gamma'_{ji}) - 2\Omega(\epsilon_{il3}\Psi_{lj} + \epsilon_{jl3}\Psi_{il}) + \chi_{ij} = 0, \tag{93}$$

$$\left(\frac{\partial}{\partial t} + a + b\right)\Gamma_{ij} - iF(\Phi_{ij} - \Psi_{ij}) - 2\Omega\epsilon_{jl3}\Gamma_{il} - \vartheta^{*\prime}_{ji} = 0, \tag{94}$$

$$\left(\frac{\partial}{\partial t} + a + b\right)\Gamma'_{ji} + iF(\Phi_{ij} - \Psi_{ij}) - 2\Omega\epsilon_{il3}\Gamma'_{jl} - \vartheta^*_{ij} = 0 \tag{95}$$

and

$$\left(\frac{\partial}{\partial t} + a + b\right)(\Gamma_{ij} - \Gamma'_{ji}) - 2iF(\Phi_{ij} - \Psi_{ij}) - 2\Omega(\epsilon_{jl3}\Gamma_{il} - \epsilon_{il3}\Gamma'_{jl}) + \vartheta_{ij} = 0, \tag{96}$$

where the abbreviations (39) have been introduced. Equation (95) has been obtained from Eq. (85) by interchange of i and j and $\mathbf{x}$ and $\mathbf{x}'$.

We shall now seek solutions of Eqs. (92) to (96) which have an exponential dependence on time of the form exp (mt). As before we shall obtain a characteristic equation for m. For the diagonal terms ($i = j$) the four first equations give

$$(m + 2a)\Phi_{ii} - iF(\Gamma_{ii} - \Gamma'_{ii}) = 0, \tag{97}$$

$$(m + 2b)\Psi_{ii} + iF(\Gamma_{ii} - \Gamma'_{ii}) - 2\Omega\epsilon_{il3}(\Psi_{li} + \Psi_{il}) = 0, \tag{98}$$

(no summation)

$$(m + a + b)\Gamma_{ii} - iF(\Phi_{ii} - \Psi_{ii}) - 2\Omega\epsilon_{il3}\Gamma_{il} = 0, \tag{99}$$

$$(m + a + b)\Gamma'_{ii} + iF(\Phi_{ii} - \Psi_{ii}) - 2\Omega\epsilon_{il3}\Gamma'_{il} = 0, \tag{100}$$

where use has been made of the relation (87). Interchanging i and j in Eq. (92) and subtracting the resulting equation from Eq. (92) we obtain

$$(m + 2a)(\Phi_{ij} - \Phi_{ji}) - iF(\Gamma_{ij} - \Gamma'_{ji} - \Gamma_{ji} + \Gamma'_{ij}) = 0, \ (i \neq j). \tag{101}$$

Similarly for Eqs. (93) and (96) we obtain

$$(m + 2b)(\Psi_{ij} - \Psi_{ji}) + iF(\Gamma_{ij} - \Gamma'_{ji} - \Gamma_{ji} + \Gamma'_{ij}) = 0 \tag{102}$$

and

$$(m + a + b)(\Gamma_{ij} - \Gamma'_{ji} - \Gamma_{ji} + \Gamma'_{ij}) - 2iF[(\Phi_{ij} - \Phi_{ji}) - (\Psi_{ij} - \Psi_{ji})]$$
$$+ 2\Omega\epsilon'_{ij3}(\Gamma_{ii} + \Gamma'_{ii} + \Gamma_{jj} + \Gamma'_{jj}) = 0, \qquad (i \neq j; \text{no summation}) \tag{103}$$

where ϵ'_{ij3} refers to off diagonal terms only and $\epsilon'_{ij3} = -\epsilon'_{ji3}$. Further, from the equations of the diagonal terms we derive

$$(m + a + b)(\Gamma_{ii} + \Gamma'_{ii} + \Gamma_{jj} + \Gamma'_{jj}) - 2\Omega\epsilon'_{ij3}(\Gamma_{ij} - \Gamma'_{ji} - \Gamma_{ji} + \Gamma'_{ij}) = 0. \qquad (i \neq j;\ \text{no summation}) \qquad (104)$$

The system of Eqs. (101), (102), (103) and (104) will be satisfied if the determinant

$$\begin{vmatrix} m + 2a & 0 & -iF & 0 \\ 0 & m + 2b & iF & 0 \\ -2iF & 2iF & m + a + b & 2\Omega\epsilon'_{ij3} \\ 0 & 0 & -2\Omega\epsilon'_{ij3} & m + a + b \end{vmatrix} \qquad (105)$$

vanishes. Hence

$$(m + a + b)^2[(m + 2a)(m + 2b) + 4F^2] + 4\Omega^2\epsilon'^2_{ij3}(m + 2a)(m + 2b) = 0. \qquad (106)$$

Letting

$$M = m + a + b \quad \text{and} \quad c = a - b, \qquad (107)$$

Eq. (106) can be written in the form

$$M^4 - M^2(c^2 - 4F^2 - 4\Omega^2\epsilon'^2_{ij3}) - 4\Omega^2\epsilon'^2_{ij3}c^2 = 0. \qquad (108)$$

The possible values of m are therefore

$$m_k = -(a + b) \pm (2)^{-1/2}\{c^2 - 4F^2 - 4\Omega^2\epsilon'^2_{ij3} \pm [(c^2 - 4F^2)^2 + 8\Omega^2\epsilon'^2_{ij3}(c^2 + 4F^2 + 2\Omega^2\epsilon'^2_{ij3})]^{1/2}\}^{1/2} \qquad (k = 4, 5, 6, 7). \qquad (109)$$

3. *Discussion of the law of decay.* It is seen from Eqs. (97), (98), (99), (100) and the result (109) that the time dependence of the spectral tensors with at least one index in the direction of the angular velocity vector is uninfluenced by the Coriolis force and the solutions have the same form as given by Eqs. (41) and (42). However, if i and j differ from 3 the results of Sec. III are modified, each tensor splitting up into four terms:

$$\Phi_{ij}(\kappa, t) = \sum_k \Phi^{(k)}(\kappa) \exp(m_k t) \qquad (k = 4, 5, 6, 7;\ i \neq 3, j \neq 3). \qquad (110)$$

We have corresponding solutions for the remaining tensors. The behaviour of the solutions in the limiting cases $W = 0$, $\Omega = 0$ and $W = \Omega = 0$ will not be discussed in detail here.

We shall now discuss the manner in which the angular velocity modifies the spectral tensors when $i \neq 3$, $j \neq 3$ and $\epsilon'^2_{ij3} = 1$. Let

$$\zeta = 2(\kappa_2 W_2 + \kappa_3 W_3)/[\kappa^2(\lambda - \nu)] \qquad (111)$$

and

$$\eta = 2\Omega/[\kappa^2(\lambda - \nu)]; \qquad (112)$$

ζ and η are associated with the forms $W_c L_c/(\lambda - \nu)$ and $U_c L_c/(\lambda - \nu)$ respectively, where W_c, L_c and U_c refer to characteristic wave velocities, lengths and rotation veloci-

ties. They may also be connected with parameters used in earlier works.[4,7,13] For small values of Ω the factors of decay become

$$m_{4,5} = -(a + b) \pm c\{1 - \zeta^2[1 - \eta^2/(1 - \zeta^2)]\}^{1/2} \tag{113}$$

and

$$m_{6,7} = -(a + b) \pm ic\eta/(1 - \zeta^2)^{1/2}, \tag{114}$$

provided $\eta^2(1 + \zeta^2)/(1 - \zeta^2) \ll 1$. From a comparison of Eqs. (109) and (41) we conclude that

$$m_{4,5} \to m_{1,2} \quad \text{and} \quad m_{6,7} \to m_3 \quad \text{when} \quad \Omega \to 0. \tag{115}$$

Thus, if the aperiodic solutions (41) are perturbed by a small angular velocity, the damping effect of the magnetic field in the factors, m_1 and m_2 will be counteracted while the term in m_3 will be split up into two periodic solutions. For sufficiently small values of Ω and F the spectral vorticity tensor will have the asymptotic behaviour

$$\Psi_{ij}(\boldsymbol{\kappa}, t) \approx \Psi_{ij}^{(4)}(\boldsymbol{\kappa}) \exp\{-2[\nu\kappa^2 + (\kappa_2 W_2 + \kappa_3 W_3)^2[1 - 4\Omega^2/\kappa^4(\lambda - \nu)^2]/\kappa^2(\lambda - \nu)]t\}$$
$$(i \neq 3, j \neq 3); \tag{116}$$

this corresponds to the earlier law (71).

The occurence of periodicity can be discussed in terms of M, given by Eqs. (107) and (109). We have

$$2M^2 = c^2(1 - \zeta^2 - \eta^2)\{1 \pm [1 + 4\eta^2/(1 - \zeta^2 - \eta^2)^2]^{1/2}\}. \tag{117}$$

If $\zeta^2 + \eta^2 < 1$ the solutions corresponding to the roots $m_{4,5}$ are aperiodic while the solutions corresponding to $m_{6,7}$ are periodic. On the other hand if $\zeta^2 + \eta^2 > 1$ the roots $m_{4,5}$ lead to periodic solutions while $m_{6,7}$ lead to aperiodic solutions. The critical case is given by

$$\zeta^2 + \eta^2 = 1, \tag{118}$$

when m_k has the values

$$m_{4,5}^{(c)} = -(a + b) \pm (2\Omega c)^{1/2}, \qquad m_{6,7}^{(c)} = -(a + b) \pm i(2\Omega c)^{1/2}. \tag{119}$$

The result (109) may be confirmed by a discussion similar to that in Sec. III, 4. Thus, reconsidering the situation indicated in Fig. 1 we shall now assume that the system is partaking in rotation and for simplicity suppose that the axis of rotation is in the z-direction. Further we shall assume a homogeneous state and that it is possible to introduce a local rectangular system of coordinates in which the approximations

$$\partial/\partial x \approx 0, \qquad \partial/\partial y \approx 0 \tag{120}$$

are valid. Equations (5) and (6) for the x- and y-components of small disturbances, $\mathbf{V}$ and $\mathbf{v}$, become

$$\left(\frac{\partial}{\partial t} - \lambda \frac{\partial^2}{\partial z^2}\right)(V_x, V_y) = W \frac{\partial}{\partial z}(v_x, v_y) \tag{121}$$

and

$$\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial z^2}\right)(v_x, v_y) = W \frac{\partial}{\partial z}(V_x, V_y) + 2\Omega(v_y, -v_x). \tag{122}$$

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[13] S. Chandrasekhar, Proc. Roy. Soc. **A216**, 293 (1953).

If V_x and V_y are eliminated we get

$$\left\{\left(\frac{\partial}{\partial t} - \lambda \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial z^2}\right) - W^2 \frac{\partial^2}{\partial z^2}\right\}(v_x , v_y) = 2\Omega\left(\frac{\partial}{\partial t} - \lambda \frac{\partial^2}{\partial z^2}\right)(v_y , -v_x). \tag{123}$$

Separating the variables in the manner

$$v_i = v_i^{(0)} \sin(\kappa z) \exp(nt), \qquad \kappa = \pi k/L \qquad (k = 1, 2, \cdots), \tag{124}$$

we find that with the abbreviations (39) we must satisfy the equation

$$[(n + a)(n + b) + F^2](v_x , v_y) = 2\Omega(n + a)(v_y , -v_x). \tag{125}$$

The condition that Eq. (125) allows non-zero solutions is

$$[(n + a)(n + b) + F^2]^2 + 4\Omega^2(n + a)^2 = 0. \tag{126}$$

In other words

$$n = -\tfrac{1}{2}[(a + b) \pm 2i\Omega] \pm \tfrac{1}{2}\{[(a - b) \pm 2i\Omega]^2 - 4F^2\}^{1/2}. \tag{127}$$

A product of two of these solutions gives exponents, $n^{(\mathrm{I})} \pm n^{(\mathrm{II})}$, which are easily seen to be of the form (109); thus

$$\begin{aligned} n^{(\mathrm{I})} \pm n^{(\mathrm{II})} &= -(a + b) \pm \tfrac{1}{2}\{(c + 2i\Omega)^2 - 4F^2\}^{1/2} \pm \tfrac{1}{2}\{(c - 2i\Omega)^2 - 4F^2\}^{1/2} \\ &= -(a + b) \pm N, \end{aligned} \tag{128}$$

where

$$N^2 = \tfrac{1}{2}(c^2 - 4\Omega^2 - 4F^2) \pm \tfrac{1}{2}[(c^2 - 4\Omega^2 - 4F^2)^2 + 16\Omega^2c^2]^{1/2}; \tag{129}$$

in other words the same values of M^2 as given by Eq. (117).[14]

The results of this section may be summed up as follows. If an angular velocity, $\mathbf{\Omega}$, is introduced into the turbulent state of motion of an electrically conducting liquid in a magnetic field the factors of decay of the tensor components *perpendicular to* $\mathbf{\Omega}$ *will be split up into four solutions. The passage from periodic to aperiodic motions is displaced* from $\zeta^2 = 1$ to $\zeta^2 < 1$, i.e., to weaker magnetic fields. For small values of Ω, the damping effect of the magnetic field in the aperiodic solutions is *counteracted by the angular velocity.* The remaining tensor components are uninfluenced by the angular velocity.

4. *The importance of the Coriolis force.* From the foregoing discussion it is seen that the effect of the Coriolis force will be of the same order of magnitude as the electromagnetic effects if ζ and η are of the same order, or from the expressions (111) and (112) if

$$W = B/(\mu\rho)^{1/2} \approx \Omega L/2\pi, \tag{130}$$

where L represents the linear dimensions of the disturbance. For the sun typical values are $W = 4\ m/s$ and $\Omega = 2.7 \times 10^{-6}\ s^{-1}$; it is seen that for these values the Coriolis force plays an essential role when the linear dimensions of the disturbance become larger than $L = 6 \times 10^6\ m$. Since the solar radius is about $7 \times 10^8\ m$ this will apply to a wide range of possible disturbances. It will probably be even more pronounced for stars with larger angular velocities.

In experiments on turbulence on laboratory scale, the largest turbulence elements

[14]The same solutions are also obtained in the non-dissipative state from results by Chandrasekhar, Astrophys. J. 119, 7 (1954).

will hardly exceed 10^{-2} m; and in mercury a common value of the wave velocity is about 5 m/s and of the angular velocity about 10 s^{-1}. Under these circumstances the electromagnetic effect will be about 50 times larger than that of the Coriolis force.

The influence of the Coriolis force on the critical damping in liquid sodium, as given by

$$(\kappa_k W_k)^2 + \Omega^2 = \kappa^4(\lambda - \nu)^2/4 \tag{131}$$

is shown in Fig. 3.

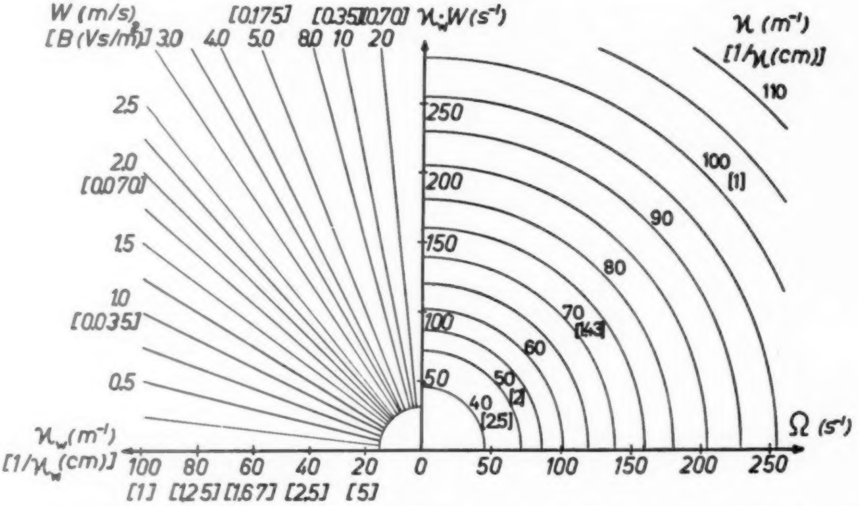

FIG. 3. Critical values of the wave velocity, W, and the angular velocity, $\mathbf{\Omega}$, in liquid sodium for turbulence elements with velocity components perpendicular to $\mathbf{\Omega}$. The total wave number is $\boldsymbol{\kappa}$ and κ_W is the component in the direction of the magnetic field, $\mathbf{B}$, where $\mathbf{W} = \mathbf{B}(\mu\rho)^{-\frac{1}{2}}$. $\lambda = 0.057$ m^2/s at 120°C.

V. The stationary state. In the previous sections we have assumed a certain spectral distribution of turbulence at an initial time, $t = 0$, and have studied the decay within a region, insulated from external sources of energy. However, in most actual cases the turbulence will be stationary and the mean intensity in every part of the spectrum will be constant with time. There is an input of energy in some parts of the spectrum and an equilibrium state is obtained, when the same amount of energy is converted into heat motion in other parts of the spectrum.

1. *The stationary equations.* We shall now investigate how far conclusions on the homogeneous, stationary state may be drawn from a linear theory that starts with the time-dependent Eqs. (31), (32), (33) and (34) for an electrically conducting liquid in a magnetic field. Introduce the tensors

$$E_{ij}^{(M)}(\mathbf{r}, t) = -\frac{\partial}{\partial t} M_{ij}(\mathbf{r}, t), \qquad E_{ij}^{(K)}(\mathbf{r}, t) = -\frac{\partial}{\partial t} K_{ij}(\mathbf{r}, t) \tag{132}$$

and

$$E_{ij}^{(MH)}(\mathbf{r}, t) = -W_k \frac{\partial}{\partial r_k} [L_{ij}(\mathbf{r}, t) - L_{ji}(-\mathbf{r}, t)] \tag{133}$$

and the corresponding Fourier transforms $\epsilon_{ij}^{(M)}(\kappa, t)$, $\epsilon_{ij}^{(K)}(\kappa, t)$ and $\epsilon_{ij}^{(MH)}(\kappa, t)$, where

$$\epsilon_{ij}^{(M)}(\kappa, t) = (8\pi^3)^{-1} \iiint E_{ij}^{(M)}(\mathbf{r}, t) \exp(-i\kappa\cdot\mathbf{r})\, dr_1\, dr_2\, dr_3 \qquad \text{etc.} \tag{134}$$

It is clear that $E_{ii}^{(M)}(0, t)$ and $E_{ii}^{(K)}(0, t)$ are the total rate of magnetic and kinetic energy losses and $E_{ii}^{(MH)}(0, t)$ is the decrease in magnetic energy per unit time and unit mass due to the magneto-hydrodynamic interaction as shown by Eqs. (29) and (30). Equations (31,) (32), (33) and (34) can now be written as

$$E_{ij}^{(M)}(\mathbf{r}, t) = E_{ij}^{(MH)}(\mathbf{r}, t) - 2\lambda\nabla^2 M_{ij}(\mathbf{r}, t), \tag{135}$$

$$E_{ij}^{(K)}(\mathbf{r}, t) = -E_{ij}^{(MH)}(\mathbf{r}, t) - 2\nu\nabla^2 K_{ij}(\mathbf{r}, t) \tag{136}$$

and

$$\left[\frac{\partial}{\partial t} - (\lambda + \nu)\nabla^2\right] E_{ij}^{(MH)}(\mathbf{r}, t) = -2W_k^2 \frac{\partial^2}{\partial r_k^2}[M_{ij}(\mathbf{r}, t) - K_{ij}(\mathbf{r}, t)]; \tag{137}$$

and the corresponding spectral equations are

$$\epsilon_{ij}^{(M)}(\kappa, t) = \epsilon_{ij}^{(MH)}(\kappa, t) + 2\lambda\kappa^2\Lambda_{ij}(\kappa, t), \tag{138}$$

$$\epsilon_{ij}^{(K)}(\kappa, t) = -\epsilon_{ij}^{(MH)}(\kappa, t) + 2\nu\kappa^2\Omega_{ij}(\kappa, t), \tag{139}$$

$$\left[\frac{\partial}{\partial t} + (\lambda + \nu)\kappa^2\right]\epsilon_{ij}^{(MH)}(\kappa, t) = 2\kappa_k^2 W_k^2[\Lambda_{ij}(\kappa, t) - \Omega_{ij}(\kappa, t)] \tag{140}$$

and

$$\epsilon_{ij}^{(MH)}(\kappa, t) = -i\kappa_k W_k[\Upsilon_{ij}(\kappa, t) - \Upsilon_{ji}(-\kappa, t)], \tag{141}$$

where the tensors of Sec. III, 2 have also been introduced.

If no external energy sources are present, the right hand sides of the contracted equations (138) and (139) represent the energies lost per unit time in the range $d\kappa_1\, d\kappa_2\, d\kappa_3$ due to interaction and dissipation. In a stationary state this loss is balanced by an equal input of energy from external sources. We now make the assumption that these considerations are valid not only for the contracted expressions at $\mathbf{r} = 0$ but can be generalized to all components and all values of $\mathbf{r}$, i.e., the stationary problem corresponds to the case when there is no dependence on t in Eqs. (133), (135), (136), (137) and their spectral equivalents. The quantities $E_{ij}^{(M)}(\mathbf{r})$ and $E_{ij}^{(K)}(\mathbf{r})$ now represent the external magnetic and kinetic "inputs of energy" per unit time.

There is no formal difficulty in extending the foregoing results to the general, non-linear case. The various correlations can be formed in the same straightforward manner as in Sec. III and the derivations need not be given here. The stationary equivalents of Eqs. (138) and (139) become

$$\epsilon_{ij}^{(M)}(\kappa) = \epsilon_{ij}^{(MH)}(\kappa) + 2\lambda\kappa^2\Lambda_{ij}(\kappa) + \alpha_{ij}(\kappa) \tag{142}$$

and

$$\epsilon_{ij}^{(K)}(\kappa) = -\epsilon_{ij}^{(MH)}(\kappa) + 2\nu\kappa^2\Omega_{ij}(\kappa) + \beta_{ij}(\kappa), \tag{143}$$

where α_{ij} and β_{ij} are the non-linear terms. The sum of these equations is

$$\epsilon_{ij}(\kappa) = 2\lambda\kappa^2\Lambda_{ij}(\kappa) + 2\nu\kappa^2\Omega_{ij}(\kappa) + \alpha_{ij}(\kappa) + \beta_{ij}(\kappa). \tag{144}$$

2. *The terms of interaction.* The equations in the preceding section form a starting point for a discussion of the physical consequences of different approximations. The simplest situation occurs for small amplitudes and without external fields when there are no interaction terms in Eqs. (142) and (143). If the external sources are taken away the turbulent fields of the magnetic and the kinetic energies decay independently of each other as is evident from Eqs. (52) and (53). In the stationary state every spectral range requires a separate magnetic and kinetic energy source; further there will be no exchange also of the energy in the different wave numbers and in different directions[15].

However, if an external magnetic field is introduced the expressions (142) and (143) show that there is now a coupling between the magnetic and the kinetic fields in every spectral range, also in first order. With an input $\epsilon_{ii}^{(M)}(\boldsymbol{\kappa}_0) = \epsilon_0^{(M)}$ of magnetic energy per unit time and unit range of wave number space and an input $\epsilon_0^{(K)}$ of kinetic energy at the wave number $\boldsymbol{\kappa}_0$ the equations (142), (143) and (140) become:

$$\epsilon_0^{(M)} = \epsilon_{ii}^{(MH)}(\boldsymbol{\kappa}_0) + 2\lambda\kappa_0^2\Lambda_{ii}(\boldsymbol{\kappa}_0), \tag{145}$$

$$\epsilon_0^{(K)} = -\epsilon_{ii}^{(MH)}(\boldsymbol{\kappa}_0) + 2\nu\kappa_0^2\Omega_{ii}(\boldsymbol{\kappa}_0) \tag{146}$$

and

$$(\lambda + \nu)\kappa_0^2\epsilon_{ii}^{(MH)}(\boldsymbol{\kappa}_0) = 2\kappa_{k0}^2W_k^2[\Lambda_{ii}(\boldsymbol{\kappa}_0) - \Omega_{ii}(\boldsymbol{\kappa}_0)]. \tag{147}$$

The solutions of these equations are

$$\Lambda_{ii}(\boldsymbol{\kappa}_0) = \{[\nu(\lambda + \nu)\kappa_0^4 + F_0^2]\epsilon_0^{(M)} + F_0^2\epsilon_0^{(K)}\}/\{2\kappa_0^2(\lambda + \nu)(\lambda\nu\kappa_0^4 + F_0^2)\},\ F_0 = \kappa_{k0}W_k \tag{148}$$

and

$$\Omega_{ii}(\boldsymbol{\kappa}_0) = \{[\lambda(\lambda + \nu)\kappa_0^4 + F_0^2]\epsilon_0^{(K)} + F_0^2\epsilon_0^{(M)}\}/\{2\kappa_0^2(\lambda + \nu)(\lambda\nu\kappa_0^4 + F_0^2)\}. \tag{149}$$

In particular for $W \to \infty$ we have

$$\Lambda_{ii}/\Omega_{ii} = 1. \tag{150}$$

This is to be expected, since the losses become less important for large values of the external field and the magneto-hydrodynamic waves approach the "ideal" state, where

$$\mathbf{V} = \mathbf{h}(\mu/\rho)^{1/2} = \mathbf{v} \tag{151}$$

and there is equipartition between the magnetic and the kinetic energies. Especially for zero input of magnetic energy, the kinetic energy given by Eq. (149) decreases with a ratio

$$\Omega_{ii}(\boldsymbol{\kappa}_0, F_0 = \infty)/\Omega_{ii}(\boldsymbol{\kappa}_0, F_0 = 0) = \nu/(\lambda + \nu) \tag{152}$$

when the quantity F_0 increases from zero to infinity. This conversion of three-dimensional turbulence into two-dimensional turbulence is consistent with the decrease in the time of decay given by Eq. (69).

Finally, we may conclude as in earlier discussions of turbulence that an energy transport through the whole spectrum is only possible in the presence of the non-linear terms. The sum of these terms in Eq. (144) generally differs from zero and there is an energy flow through the spectral range in question.

[15]A discussion of the properties of the interaction terms in hydrodynamics may be found in G. K. Batchelor, *The theory of homogeneous turbulence*, Cambridge Univ. Press, Chap. V, 1953.

VI. Applications to experiments. Experiments on turbulent flow of mercury in a homogeneous magnetic field have been carried out by Hartmann[5,6] and Lehnert[7,8] with a maximum field strength of about 1 Vs/m^2 (10^4 gauss). Hartmann has shown that the necessary pressure difference to keep constant volume flow between the ends of a rectangular channel decreases in the turbulent state when the magnetic field increases (similar investigations have been carried out by Murgatroyd[16]). In the latter experiment the torque between an inner, rotating and an outer, fixed cylinder decreases for a given angular velocity if a homogeneous, axial magnetic field is introduced. Thus, in both experiments the field seems to have the effect of suppressing the intensity of turbulence.

These results may be given a qualitative explanation in terms of the linear theories of Secs. III and V. The energy sinks formed by the dominating dissipation for large wave numbers will cause a mean transfer of energy from the largest whirls, which receive the greatest part of the external input of energy. It is reasonable to assume that a great part of this energy will reach the energy consuming region for large wave numbers before being dissipated. In this region we are allowed to apply the linear theory and obtain a connexion between input power and turbulent intensity of the small scale motion. Since it is unlikely that the external magnetic field will have a reverse effect on the large scale motion we may expect that the linear theory would not depart too far from the real physical situation.

From Eq. (71) and Fig. 2, giving the result of the theory of decay of turbulence, the damping effect on vortices having finite wave numbers in the direction of the magnetic field the reduction of the total isotropic intensity by a factor of about 2/3 is apparent. This implies that a given intensity of the kinetic energy Ω_{ii} requires a larger input of energy in the presence of the field than if the field were absent. Further, the tangential force due to turbulent exchange of momentum to the walls of the experimental arrangements increases with the intensity of the kinetic energy. Thus, a given pressure difference or a given torque requires an increasing input of energy for an increasing field strength, i.e. an increasing volume flow and an increasing angular velocity respectively. Consequently, if a magnetic field is present, the curves for the pressure difference and the torque as functions of the volume flow and the angular velocity respectively must lie below the corresponding curves in the absence of the magnetic field.

These same conclusions may be drawn from Eqs. (149) and (152), representing stationary turbulence; these equations show how the intensity of the kinetic energy decreases with increasing field and constant input of energy. Even if we assume an input of both kinds of energy the ratio

$$\Omega_{ii}(\kappa_0\,, F_0 = \infty)/\Omega_{ii}(\kappa_0\,, F_0 = 0) = \nu\epsilon_0^{(K)}/[(\lambda + \nu)(\epsilon_0^{(K)} + \epsilon_0^{(M)})] \tag{153}$$

will be very small, since $\lambda \gg \nu$ for mercury and $E^{(K)}$ and $E^{(M)}$ will probably be of the same order during the transport of energy towards high wave numbers. Consequently the conclusions both from the theory of decay and the stationary theory of turbulence are in agreement with experimental observations.

Finally, it also follows from the results of Sec. IV, 4 that the Coriolis force does not play any essential role in the experiment with rotating cylinders.

VII. Concluding remarks. A linear theory gives some information about the physical nature of turbulence in magneto-hydrodynamics, especially in the case of linear inter-

[16] Phil. Mag. **44**, 1348 (1953).

action forces caused by external magnetic fields and Coriolis fields. However, this does not show the feature of the general turbulent interaction and the process of distribution of energy. One way to solve this difficult problem is to continue the work on the "quasi-statistical" theory of correlations. Another way, which at the present stage also seems to be associated with great difficulties, is to make a purely statistical attack, somewhat along the lines of Burgers[17] and Onsager[18]. But such a theory, even in the absence of external fields, is probably not available.

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A NEW SINGULARITY OF TRANSONIC PLANE FLOWS*

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Introduction. The limit line singularity of the hodograph method [1,2,3] cannot arise in transonic plane flows past a physical boundary of finite curvature [4, 5, 6, 7]. In this note we discuss a new singularity of the hodograph method which gives infinite acceleration of the fluid but avoids the difficulties of the limit line theory.

1. On the non-occurrence of the limit line in transonic plane flow. We follow Morawetz and Kolodner [7] in regard to notation and write the hodograph equations as

$$\varphi_\theta = \rho^{-1} q \psi_q , \tag{1.1}$$

$$\varphi_q = \rho^{-1} q^{-1} c^{-2} [q^2 - c^2] \psi_\theta . \tag{1.2}$$

Here $\varphi(q, \theta)$ is the potential function, $\psi(q, \theta)$ the stream function, ρ is a known function of q and $c^2 = dp/d\rho$ where $p = p(\rho)$ is the equation of state.

We define characteristic variables α and β by

$$2\, d\alpha = \rho q^{-1} k \, dq - d\theta \equiv d\sigma - d\theta, \tag{1.3}$$

$$2\, d\beta = \rho q^{-1} k \, dq + d\theta \equiv d\sigma + d\theta, \tag{1.4}$$

say, where

$$k^2 = \rho^{-2} c^{-2} (q^2 - c^2), \tag{1.5}$$

so that $k = k\,(\alpha + \beta)$, and k increases with q.

Then (1.1) and (1.2) may be written

$$\varphi_\alpha = -k \psi_\alpha , \tag{1.6}$$

$$\varphi_\beta = k \psi_\beta . \tag{1.7}$$

The limit line is defined by the locus $J = 0$, where

$$J = \frac{\partial(x, y)}{\partial(\theta, q)} = \rho^{-2} q^{-1} [\psi_q^2 - k^2 \rho^2 q^{-2} \psi_\theta^2] = q^{-3} k^2 \psi_\alpha \psi_\beta . \tag{1.8}$$

By (1.6) and (1.7), ψ satisfies

$$2k \psi_{\alpha\beta} + k_\alpha (\psi_\alpha + \psi_\beta) = 0. \tag{1.9}$$

We shall prove that no limit lines occur if:

(a) the physical boundary has finite curvature, say $K = K(\theta)$, and

(b) the hodograph image of this boundary has a continuously turning tangent which is never in the characteristic direction, say $\beta = \beta(\alpha)$ with $\beta'(\alpha)$ finite and non-zero for the equation of the hodograph boundary.

We first show that ψ_q and ψ_θ , and therefore J, are bounded in the supersonic region

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and on the sonic line. As in [7] we have

$$K(\theta) = -(\rho q J)^{-1}\psi_q . \tag{1.10}$$

Combining Eqs. (1.3), (1.4), (1.8) and (1.10), with $\psi_\alpha/\psi_\beta = -\beta'(\alpha)$ we see that, given $K(\theta)$ and $\beta(\alpha)$, both ψ_α and ψ_β take determinate finite values at each point of the hodograph boundary, i.e. knowledge of the physical boundary and of its hodograph image leads to Cauchy data for the hodograph solution $\psi(\alpha, \beta)$ which is thereby uniquely determined in the supersonic region [6]. Clearly ψ_α , and ψ_β are finite at all *supersonic* points since any singularity would be propagated along at least one characteristic to the boundary where we know that ψ_α , ψ_β are finite. For sonic points we take the origin of α, β on the sonic line and note that $k \sim (\alpha + \beta)^{1/3}$. (We use the sign $\sim$ to denote the *form* of the dominant term in an expansion near a singularity and the sign $\cong$ when the numerical factor is significant). The dominant terms in the solutions of Eq. (1.10) must therefore satisfy

$$6(\alpha + \beta)\psi_{\alpha\beta} + \psi_\alpha + \psi_\beta = 0, \tag{1.11}$$

which is homogeneous in α, β while the terms neglected are of a higher order. Now ψ_α , ψ_β are finite at all supersonic points near F, so the theory of Darboux [8, p. 90] applies as α, β tend to zero.

In our notation

$$\psi(\alpha, \beta) \sim (\alpha + \beta)^{2/3}g(\alpha, \beta) + h(\alpha, \beta), \tag{1.12}$$

where g and h are regular.

We see at once that $\psi_\theta \sim \psi_\beta - \psi_\alpha$ and $\psi_q \sim (\alpha + \beta)^{1/3}(\psi_\alpha + \psi_\beta)$ are bounded on the sonic line. Writing U for the values of $k\psi_\alpha$ on the characteristic line $\alpha = \text{const.}$, Eq. (1.9) gives

$$dU/dk = -\psi_\theta . \tag{1.13}$$

From (1.8) and (1.13) and for values on $\alpha = \text{const.}$

$$J = -q^3k^2(d/dk)(U^2k^{-1}). \tag{1.14}$$

If U vanishes for $k = k_0$ and ψ_θ is bounded, say $|\psi_\theta| < M$, Eq. (1.13) gives $|U| < M(k - k_0)$ so that U^2k^{-1} certainly vanishes at $k = k_0$. The case $k_0 = 0$, i.e. where the zero is on the sonic line is included in this argument. Since U^2k^{-1} is positive near $k = k_0$ we deduce that it is increasing near $k = k_0$ and so the derivative in (1.14) is positive. We see then that J changes sign from positive to negative with k and q increasing and a similar argument would show that if $V = k\psi_\beta$ vanishes J changes sign along the characteristic $\beta = \text{const.}$

Now conditions (a) and (b) also ensure that $J > 0$ on the boundary and the reasoning of Manwell [6] and Morawetz and Kolodner [7] can be applied to complete the proof that J is positive throughout the region.

2. A singularity induced by a corner on the hodograph boundary. We now relax condition (b) of Sec. 1 by admitting a single corner at point C on the hodograph boundary, (see Fig. 1). The solution $\psi(\alpha, \beta)$ is still uniquely defined by giving $\beta = \beta(\alpha)$ and $K = K(\theta)$ but the solution has discontinuities in the first derivatives with respect to α and β.

Observing that the leading terms in the expansions near C must satisfy $\psi_{\alpha\beta} = 0$ we

find without difficulty that the solutions in regions I and II may be written

$$\psi_1 \cong d[\alpha(1 + \epsilon)/(1 - \epsilon) + \beta], \tag{2.1}$$

$$\psi_2 \cong d[\alpha(1 + \epsilon)/(1 + \epsilon') + \beta], \tag{2.2}$$

where the tangents at C are $\sigma = \epsilon\,\theta$ and $\sigma = -\epsilon'\theta$ while d is a constant and the origin is at point C.

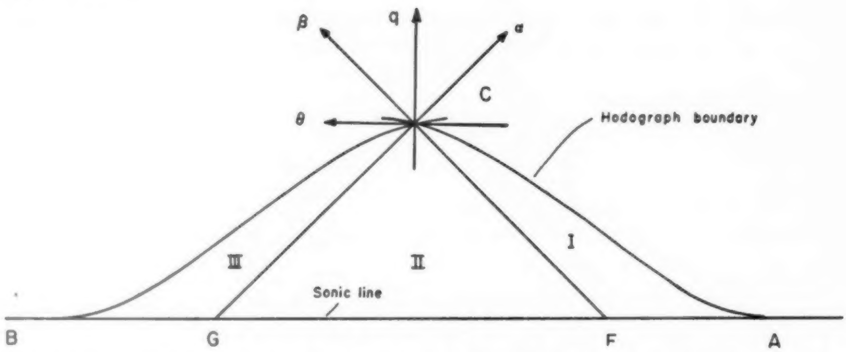

FIG. 1. Hodgraph Plane, Supersonic Region in Characteristic Coordinates α, β.

The discontinuity in ψ_α, say $[\psi_\alpha]$, is, see [9, p. 54] and Eq. (1.9) above, propagated along the characteristic $\alpha = 0$ according to

$$2k[\psi_\alpha]_\beta + k_\beta[\psi_\alpha] = 0, \tag{2.3}$$

which gives, with C another constant,

$$[\psi_\alpha] = Ck^{-1/2}. \tag{2.4}$$

Since ψ_β is continuous at $\alpha = 0$

$$[\psi_\theta] = -\tfrac{1}{2}Ck^{-1/2}. \tag{2.5}$$

Now the arguments of Sec. 1 still apply to the solution ψ_1 which is determined by finite data on the smooth curve AC independently of how AC is continued beyond C. In particular $(\psi_1)_\theta$, $(\psi_1)_q$ and J_1 are finite along CF. Using Eqs. (2.4), (2.5) and (1.8) we find without difficulty that, at F, $(\psi_1)_q = (\psi_2)_q$ and $J_2 = J_1$, which is non-zero according to Sec. 1, but $(\psi_2)_\theta$ tends to infinity near F with $k^{-1/2}$. According to Eq. (1.10) the streamline curvature remains finite at F but for the fluid acceleration, say f, we have $f_2 = q\,dq/ds = q(dq/d\theta)_{\psi=\text{const.}}\,(d\theta/ds) = -(qK\psi_\theta/\psi_q)_2$ which tends to infinity with ψ_θ.

In the limit line singularity both K and f become infinite together, a difficulty which is avoided here.

3. Expansion of the solution near the singularity. We write $[\psi]$ for $\psi_2 - \psi_{1,(2)}$ and set $[\psi] = 0$ in region I while $\psi_{1,(2)}$ may be any flow past a smooth profile and determined in region II by data on AC and AC slightly produced beyond C with no discontinuity in slope at C. Taking the origin at F we construct the leading term in the expansion of a solution of Eq. (1.9) as a homogeneous function of α and β satisfying Eq. (1.11), viz.,

$$[\psi] \sim \alpha\beta^{-1/6}F(1/6,\, 7/6,\, 2,\, -\alpha\beta^{-1}), \tag{3.1}$$

for $0 \leq -\alpha\beta^{-1} \leq 1$ and with $F(a, b, c, z)$ denoting the hypergeometric series (see [10, Chap. 10]).

For $\alpha = 0$ we have $[\psi] \equiv 0$ and, in conformity with Eq. (2.4),

$$[\psi_\alpha] \sim \beta^{-1/6} \sim k^{-1/2}. \tag{3.2}$$

Now $[\psi]$ is the only homogeneous solution satisfying this requirement and regular in $0 \leq -\alpha\beta^{-1} \leq 1$. For $|\alpha\beta^{-1}| < 1$ Eq. (3.1) may be differentiated term by term and we find that $[\psi]_q \sim \theta^{1/6}$ and $[\psi]_\theta \sim \theta^{-1/6}$. For almost all points in the hodograph plane near F, $z = -\alpha\beta^{-1}$ is nearly unity and we use the identity given by Copson [10, p. 252] yielding

$$[\psi] \sim \alpha\beta^{-1/6}[PF(1/6, 7/6, 1/3, 1 + \alpha\beta^{-1}) + Q(1 + \alpha\beta^{-1})^{2/3}F(11/6, 5/6, 5/3. 1 + \alpha\beta^{-1})], \tag{3.3}$$

where P and Q are certain numerical constants.

Writing $q = c_* + \lambda t$ we choose the constant λ so that $\sigma = t^{3/2}$ and for all finite values of t/θ

$$[\psi] \cong \theta^{5/6}[P + 2^{2/3}Qt\theta^{-2/3}]. \tag{3.4}$$

We can show too that Eq. (3.4) may be differentiated giving

$$[\psi]_\theta = (5/6)P\theta^{-1/6}, \tag{3.5}$$

$$[\psi]_t = 2^{2/3}Q\theta^{1/6}. \tag{3.6}$$

From these results for the derivatives of $[\psi]$ it follows that $J_2 - J_{1.(2)}$ is uniformly small and that in spite of the infinity in acceleration at F the physical coordinates near F are changed very little by adding $[\psi]$ to $\psi_{1.(2)}$. Although there is a discontinuity in ψ_α along FC the physical solutions join up smoothly because, according to (1.6) and (1.7), not only ψ but φ is continuous between I and II.

4. The continuation of the solution across the sonic line. In general, of course, we may not joint up piecewise differentiable hodograph solutions, for the corresponding physical planes would not join up. In particular we require continuity of both ψ and ψ_q at the sonic line (see Fig. 2). We write

$$\theta = s \cos \Theta, \qquad |t|^{3/2} = s \sin \Theta, \tag{4.1}$$

with $0 \leq \Theta \leq \pi$. Suitable solutions are found as the real and imaginary parts of

$$\Psi \sim s^{5/6} \exp [7i(\pi - \Theta)/6]F(1/6, 7/6, 2, \exp [-2i\Theta]), \tag{4.2}$$

and two further solutions are found by replacing F by

$$G = F \int Z^{-2}F^{-2}(1 - Z)^{-1/3}\, dZ \tag{4.3}$$

where $Z = \exp(-2i\Theta)$.

Except in the vicinity of $\theta = 0$ we have either Θ nearly zero or Θ nearly π and so Z is nearly unity in either case. Using Eq. (3.3) for complex values of Z and Eqs. (4.2), (4.3) it is not difficult to show that, except for θ nearly zero, the most general complex Ψ is given by

$$\Psi \sim \theta^{5/6} \exp [7i(\pi - \Theta)/6][R + S\,|\sin \Theta|^{2/3} \exp [i(\pi - 2\Theta)/3], \tag{4.4}$$

where R and S are arbitrary constants.

Taking S zero we find a solution

$$\Psi_1 \cong \theta^{5/6} \sin [7(\pi - \Theta)/6], \tag{4.5}$$

and similarly making R zero we get

$$\Psi_2 = \theta^{1/6} \mid t \mid \sin [11(\pi - \Theta)/6]. \tag{4.6}$$

Once more it can be shown that these expressions may be differentiated with respect to t or θ and easily $d\Theta/dt = 0$ for $t = 0$ so both derivatives vanish on $\Theta = \pi$, i.e. $\Psi_\theta = \Psi_q = 0$ for $\theta < 0$. We form the combination

$$[\Psi] = 2P\Psi_1 + 2^{5/3}Q\Psi_2 \tag{4.7}$$

and verify that it satisfies both $\Psi = \Psi_q = 0$ for $\mid t \mid = 0$ and $\theta < 0$ and conditions (3.5) and (3.6) for $\mid t \mid = 0$ and $\theta > 0$.

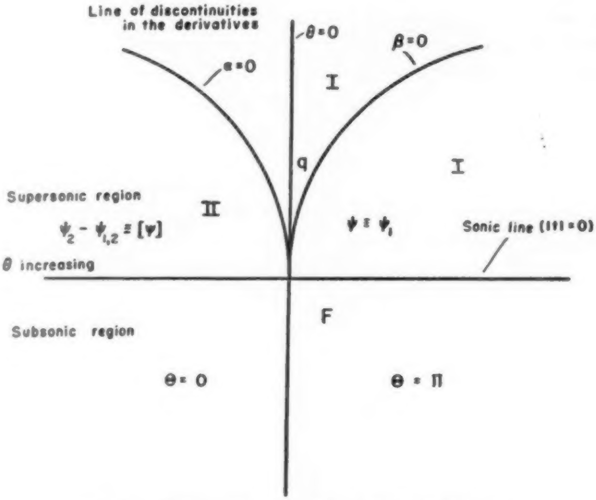

FIG. 2. Hodograph Plane Near Critical Point F.

A simple discussion shows that the other two linearly independent solutions given by Eq. (4.4) cannot occur and so $[\Psi]$ which was uniquely determined in region II, save for a constant multiplier depending on data near C, has a unique continuation across the sonic line, viz. $[\Psi]$ of Eq. (4.7). It is easily verified that for any Ψ and near $\theta = 0$ we have $\Psi_\theta \sim \mid t \mid^{-1/4}$ and $\Psi_t \sim \mid t \mid^{-1/4}$. (We use Eqs. (4.2) and (4.3) and prove that $F(Z)$ has no roots on $\mid Z \mid = 1$). Noting also Eqs. (4.5) and (4.6) we see that Ψ has finite derivatives at all subsonic points near F. The solution in the physical plane can be constructed by adding $[\Psi]$ to the continuation across $\mid t \mid = 0$ of the function ψ_1 and performing the usual integrations.

The discussion of Secs. 2, 3, 4 applies with obvious changes to the propagation of discontinuities along the other characteristic CG and a singularity develops similarly at G (Fig. 1).

5. The occurrence of the singularity in flow past a given profile. According to Sec. 1, transonic flows can be constructed in the neighborhood of an arbitrary convex physical boundary of finite curvature along which the velocity is determined by a smooth hodograph image. By Sec. 2, however, there are infinitely many adjacent velocity distributions for which the flow must break down at the sonic line, viz. those for which the hodograph images have any discontinuity in slope at supersonic points.

We turn now to the possibility that the singularity described above may arise for a given body in a subsonic stream as the Mach number of the stream is slowly increased. It must be admitted that the existence and uniqueness of plane transonic flow solutions in this situation is by no means certain. As for uniqueness, the author has proved [11] that the solutions for cyclic flow round a circular cylinder are not unique at an enumerable infinity of values of the circulation parameter and that these values lie close together. For non-circulatory flows past a precisely defined body it is possible that existence of the solutions will fail at certain (perhaps 'almost all') Mach numbers with transonic flow conditions (c.f. Frankl [12], although the argument of [12] is by no means complete).

On the other hand in a number of approximations worked out by the present writer it seems clear that the general consequence of increasing the strength of the stream for an approximately determined body is to give velocity distributions whose hodograph boundaries develop a corner at the point of maximum velocity in the supersonic region.

An example of this process is furnished by [13] which although, as appears on closer examination, merely an *approximate* solution of the problem implied by the title is a useful contribution to the problem of the behaviour of transonic flow when the boundary is kept fixed while the stream conditions are varied. (The solution of [13] satisfies correct boundary conditions on the arc BD and on the axis AB, (Fig. 3) and the flow becomes

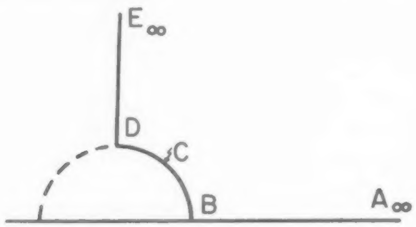

FIG. 3.

uniform at great distances but, owing to lack of *analytic continuation* the velocity vector along DE need not be parallel to AB except at D and E.)

According to [13] the image of the boundary is determined in the hodograph plane by an equation of the form

$$\cos \theta = \lambda(q)/\lambda(q_1), \tag{5.1}$$

where q_1 is the maximum velocity on the boundary. If $q_1 < q_2$, where $\lambda'(q_2) = 0$, this equation may be expanded as

$$q - q_1 \cong -\tfrac{1}{2}[\lambda'(q_1)]^{-1}\lambda(q_1)\theta^2. \tag{5.2}$$

If however $q_1 = q_2$ the expansion must be written as

$$\lambda''(q_2)(q - q_2)^2 = -\lambda(q_2)\theta^2. \tag{5.3}$$

For the *semi-circular* body which has in fact been treated in [13] it was found that

$$\lambda \cong q/c - (5/24)(q/c)^3 - [125/3456 + 35(\gamma - 1)/576](q/c)^5, \tag{5.4}$$

where the notation is that of Courant and Friedrichs [9]. To this approximation it appears that $\lambda'(q)$ vanishes for $q_2 \cong 1.03\, c$ and that the branching streamlines in the hodograph plane make an angle arc tan $(qd\theta/dq) = 1.86$ with the line $\theta = 0$.

In this example the singularity develops for only a very small supersonic region but in other cases of transonic flow worked out by the author to the same degree of approximation the supersonic region appears to develop considerably before the flow breaks down.

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CANONICAL EQUATIONS FOR NON-LINEARIZED STEADY IRROTATIONAL CONICAL FLOW*

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1. Introduction. A steady flow is *conical* if it contains a *vertex*, P, such that the velocity components, pressure, density, and entropy are constant on every half line emanating from P. Numerous authors [e.g. 5, 9, 11, 12, 19] have investigated *linearized* conical flows, which can be attacked with the aid of complex function theory inside the Mach cone of the vertex. A few [2, 16, 22] have considered second order approximations or perturbed axisymmetric non-linearized conical flow [20]. So far, the study of *non-linearized* conical flow has been confined to special examples [3, 4, 10, 13, 21], or the use of relaxation methods [14] or three-dimensional characteristic treatments [15]. It is natural to ask how far the discussion of the non-linearized case can be carried. Although no attempt has been made in this note to reach the ideal conclusion, an existence and uniqueness theorem, the present inquiry does lead at least to the formulation of an interesting boundary value problem and to a basis for numerical computation.

Most of the discussion is concerned with the flow about a conical body completely surrounded by a conical shock. The change in entropy at a shock wave is of higher order than the changes in the other flow functions. For this reason the equations of supersonic flow are frequently simplified by neglecting the variation of entropy behind the shock, as will be done here. Now the flow field *may* be but is not necessarily irrotational. In this note the flow will be assumed to be irrotational. This makes it possible to reduce the partial differential equations to simple canonical forms, to determine the class of transformations under which these are invariant, and eventually to map the conical surfaces of obstacle and shock onto *known* curves. Thus the difficulties which arise from not knowing the location of the shock wave can be avoided. Finally, it should be remarked that Taylor-Maccoll flow has all of the assumed properties for the completely elliptic case. Accordingly, this special example is a fruitful source of conjectures as to the influence of the various parameters and of estimates for numerical computation.

2. Fundamental equations. The velocity potential function, ϕ, of a steady irrotational flow satisfies

$$(a^2\delta_{ij} - u_iu_j)\partial^2\phi/\partial x_i\,\partial x_j = 0, \qquad i, j = 1, 2, 3, \tag{2.1}$$

where the velocity

$$u_i = \partial\phi/\partial x_i\,, \tag{2.2}$$

the squared velocity of sound

$$a^2 = \tfrac{1}{2}(\gamma - 1)(1 - u_iu_i), \tag{2.3}$$

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x_i are cartesian coordinates, Kronecker's $\delta_{ij} = 1(0)$ accordingly as $i = (\neq)j$, and repetition of indices implies summation over their ranges. In a conical flow with vertex (0, 0, 0) let

$$\phi = z\Phi(X), \qquad X_\alpha = x_\alpha/z, \qquad z = x_3\,, \qquad \alpha = 1, 2. \tag{2.4}$$

Note that points in the X_1X_2 plane correspond to lines through the origin of the $x_1x_2x_3$-space, and conversely. Now (2.1) becomes

$$[a^2(\delta_{\alpha\beta} + X_\alpha X_\beta) - (u_\alpha - wX_\alpha)(u_\beta - wX_\beta)]\, \partial^2\Phi/\partial X_\alpha\, \partial X_\beta = 0, \tag{2.5}$$

where

$$u_\alpha = \partial\Phi/\partial X_\alpha\,, \tag{2.6}$$

$$w \equiv u_3 = \Phi - u_\alpha X_\alpha\,. \tag{2.7}$$

If $u_\alpha(X)$ are functionally independent, subject (2.5) to the Legendre transformation (2.7) to obtain [4]

$$[a^2(\delta_{\alpha\beta} + w_\alpha w_\beta) - (u_\alpha + ww_\alpha)(u_\beta + ww_\beta)](-1)^{\alpha+\beta}\, \partial^2 w/\partial u_{\alpha+1}\, \partial u_{\beta+1} = 0, \tag{2.8}$$

where $\alpha + 1$ and $\beta + 1$ are to be reduced mod 2, and where the velocity $u_1\,, u_2\,, w(u_1\,, u_2)$ is assigned to the point

$$X_\alpha = -\partial w/\partial u_\alpha = -w_\alpha\,. \tag{2.9}$$

Now introduce more general parameters $\mu_\alpha = \mu_\alpha(u_1\,, u_2)$. Then

$$u_i = u_i(\mu_1\,, \mu_2). \tag{2.10}$$

Let

$$\begin{gathered} q^2 = u_iu_i\,, \qquad q\, \partial q/\partial\mu_\alpha = u_i\, \partial u_i/\partial\mu_\alpha\,, \qquad g_{\alpha\beta} = (\partial u_i/\partial\mu_\alpha)(\partial u_i/\partial\mu_\beta), \\ b_{\alpha\beta} = (g_{11}g_{22} - g_{12}^2)^{-1/2} \det \|\, \partial^2 u_i/\partial\mu_\alpha\, \partial\mu_\beta\,, \partial u_i/\partial\mu_1\,, \partial u_i/\partial\mu_2 \,\|. \end{gathered} \tag{2.11}$$

For the special choice $\mu_\alpha = u_\alpha$ (2.8) implies

$$(a^2 g_{\alpha\beta} - q^2\, \partial q/\partial\mu_\alpha\, \partial q/\partial\mu_\beta)(-1)^{\alpha+\beta} b_{\alpha+1\,\beta+1} = 0. \tag{2.12}$$

Since $g_{\alpha\beta}\,, b_{\alpha\beta}\,,$ and $\partial q/\partial\mu_\alpha$ are covariant tensors, (2.12) is a tensor equation. Hence its form is independent of the particular choice of parameters μ_α. Now observe that (2.9) implies $x_i\partial u_i/\partial\mu_\alpha = 0$. Thus the point $u_i(\mu)$ on the hodograph surface corresponds to the half line

$$x_i = r\nu_i\,, \qquad \nu_i\, \partial u_i/\partial\mu_\alpha = 0, \tag{2.13}$$

where $\nu_i(\mu)$ is a vector normal to this surface at $u_i(\mu)$, and r is a parameter.

So far the three functions $u_i(\mu)$ have been subjected to (2.12) only. Two more equations are required to determine u_i. As suggested by [6, p. 130], they can be chosen to simplify the coefficients of (2.12) in a way which depends on the nature of the quadratic form

$$(a^2 g_{\alpha\beta} - q^2\, \partial q/\partial\mu_\alpha\, \partial q/\partial\mu_\beta)L_\alpha L_\beta\,. \tag{2.14}$$

This is *definite* (*indefinite*) if and only if the determinant of its coefficients is greater (less) than zero. For $\mu_\alpha = u_\alpha$ and by (2.9) the determinant becomes $a^2[(1 + X_\alpha X_\alpha)(a^2 -$

$q^2) + (u_\alpha X_\alpha + w)^2]$. In *subsonic* flow (2.14) is always definite. In the *supersonic* case (2.14) will be definite (indefinite) if and only if the half line (2.13) lies *inside* (*outside*) the Mach cone based on $u_i(\mu)$ and with vertex at (0, 0, 0). In the former (latter) case equations (2.5), (2.8) and (2.12) will be said to be of *elliptic* (*hyperbolic*) *type* for reasons which will become apparent in Sec. 3 and 4.

3. The hyperbolic case. This section has been included to round out the discussion, for the sake of contrast with the material in Sec. 4. It is not essential for the sequel. Examples of conical flows which contain regions where $x_i = r\nu_i$ is outside its Mach cone have been discussed in [10].

If (2.14) is indefinite, choose the parameters μ_1 and μ_2 so that

$$A \equiv a^2 u_{i1} u_{i1} - (u_i u_{i1})^2 = 0, \tag{3.1}$$

$$B \equiv a^2 u_{i2} u_{i2} - (u_i u_{i2})^2 = 0, \tag{3.2}$$

where $u_{i\alpha} = \partial u_i / \partial \mu_\alpha$. By (2.11) and (2.12)

$$u_{i12} = C_\beta u_{i\beta} \tag{3.3}$$

for some functions C_β . Eliminate u_{i12} from $A_2 = B_1 = 0$ to obtain

$$(a^2 g_{\alpha\beta} - q^2 q_\alpha q_\beta) C_\beta = q q_\alpha g_{12} - a a_{\alpha+1} g_{\alpha\alpha} \qquad (\alpha \textit{ not} \text{ summed}). \tag{3.4}$$

Conversely, suppose $u_i(\mu)$ is a solution of (3.3), where C_β are defined by (3.4). Then $A_2 = B_1 = 0$, so

$$A = A(\mu_1), \qquad B = B(\mu_2). \tag{3.5}$$

Equation (3.3) is a *hyperbolic* system of second order quasi-linear partial differential equations with the same principal parts, and μ_1 and μ_2 are *characteristic* variables. The existence and uniqueness, in the small, of solutions of non-characteristic and characteristic initial value problems for such systems have been discussed in [6, pp. 316-326] and there are numerous references in [1]. In conical flow the initial conditions of fluid mechanical origin, discussed in Sec. 4, appear to be more complicated than those ordinarily considered in the literature. To these must be adjoined an initial condition such as

$$A = B. \tag{3.6}$$

By (3.5) this implies $A = B =$ constant. Then it will suffice to impose $A = 0$ at *one* point, to obtain $A = B = 0$.

To simplify the formulation of initial value problems, it would be desirable to standardize initial curves in the $\mu_1\mu_2$-plane as far as possible. This raises the question as to what class of transformations $\mu_\alpha = \mu_\alpha(\mu^*)$ will preserve the forms of (3.1) and (3.2). The coefficient tensor for the transform of (2.12) is

$$a^{*2} g^*_{\alpha\beta} - q^{*2} q^*_\alpha q^*_\beta = (a^2 g_{\sigma\delta} - q^2 q_\sigma q_\delta)(\partial\mu_\sigma/\partial\mu^*_\alpha)(\partial\mu_\delta/\partial\mu^*_\beta). \tag{3.7}$$

By (3.1) and (3.2) its diagonal elements vanish if and only if $(\partial\mu_1/\partial\mu^*_1)\partial\mu_2/\partial\mu^*_1 = (\partial\mu_1/\partial\mu^*_2)\partial\mu_2/\partial\mu^*_2 = 0$. Hence either $\mu_1 = \mu_1(\mu^*_1)$, $\mu_2 = \mu_2(\mu^*_2)$, or else $\mu_1 = \mu_1(\mu^*_2)$, $\mu_2 = \mu_2(\mu^*_1)$. That is, *the most general transformation with the desired invariance consists of changes of scales along the axes, possibly combined with reflection with respect to* $\mu_2 = \mu_1$. By such transformations two initial curves can be mapped simultaneously onto straight lines.

Examples such as those considered in [10, 15] make it seem likely that every conical

flow which contains regions where the equations are of hyperbolic type also contain regions of elliptic type. Hence the canonical form (3.3) cannot be used for an entire flow.

4. The elliptic case. First it should be remarked that at all points in a Taylor-Maccoll flow the ray $r\nu_i$ is inside its Mach cones. Hence (2.14) is definite throughout. It seems reasonable to conjecture that this will also be the case at least for slightly yawing or slightly distorted circular cones.

If (2.14) is definite, choose *isothermic* (relative to the tensor $a^2g_{\alpha\beta} - q^2q_\alpha q_\beta$) parameters μ_1 and μ_2, i.e. set

$$A \equiv a^2u_{i1}u_{i2} - u_iu_{i1}u_ju_{j2} = 0, \tag{4.1}$$

$$2B \equiv a^2(u_{i1}u_{i1} - u_{i2}u_{i2}) - (u_iu_{i1})^2 + (u_iu_{i2})^2 = 0. \tag{4.2}$$

Then by (2.11) and (2.12)

$$u_{i\alpha\alpha} = C_\beta u_{i\beta} \tag{4.3}$$

for some functions C_β. Now form

$$\partial A/\partial\mu_1 - \partial B/\partial\mu_2 = 0, \qquad \partial A/\partial\mu_2 + \partial B/\partial\mu_1 = 0 \tag{4.4}$$

and eliminate $u_{i\alpha\alpha}$ to obtain

$$(a^2g_{\alpha\beta} - q^2q_\alpha q_\beta)C_\beta = \tfrac{1}{2}g_{\beta\beta}(a^2 + q^2)_\alpha - g_{\alpha\beta}a^2_\beta\,. \tag{4.5}$$

Conversely, suppose $u_i(\mu)$ is a solution of (4.3), where C_β are defined by (4.5). Since this implies (4.4), $A + iB$ is an analytic function of $\mu_1 + i\mu_2$.

Equation (4.3) is a system of *elliptic* second order quasi-linear partial differential equations with the same principal parts. Since the coefficients C_β are rational functions of their nine arguments u_i, u_{i1}, and u_{i2}, the solutions of (4.3) are analytic [6, p. 339]. The existence and uniqueness of the solutions of Dirichlet's problem for such a system for a small enough region of the $\mu_1\mu_2$-plane have been discussed in [6, p. 287]. Unfortunately, the conical flow problem leads to the non-linear boundary conditions described below, rather than to the linear Dirichlet, Neumann, or mixed type usually considered.

As a preliminary to the formulation of a boundary value problem, first determine the class of transformations $\mu_\alpha = \mu_\alpha(\mu^*)$ that will preserve the forms of (4.1) and (4.2), and therefore of (4.3) and (4.5). The transformed coefficient tensor (3.7) becomes $a^{*2}g^*_{\alpha\beta} - q^{*2}q^*_\alpha q^*_\beta = [a^2g_{11} - (qq_1)^2](\partial\mu_\alpha/\partial\mu^*_\alpha)(\partial\mu_\alpha/\partial\mu^*_\beta)$. Then μ^*_α will also be isothermic parameters if and only if $(\partial\mu_\alpha/\partial\mu^*_\alpha)(\partial\mu_\alpha/\partial\mu^*_\beta) = D^2\delta_{\alpha\beta}$ for some function D. Hence, *any conformal map of the $\mu_1\mu_2$-plane onto the $\mu^*_1\mu^*_2$-plane has the desired invariance.*

Suppose that the conical body (shock) is described by $f(X_1, X_2) = 0(g(X_1, X_2) = 0)$. Assume that $f = 0$ and $g = 0$ are closed curves in the X_1X_2-plane which do not intersect themselves or each other, and that $f = 0$ is inside $g = 0$. Suppose that their images $F(\mu) = 0$ and $G(\mu) = 0$ in the $\mu_1\mu_2$-plane have the same properties. All of these conditions can be satisfied by Taylor-Maccoll flow, and should also be attainable in slightly perturbed Taylor-Maccoll flows, at least. Then the region between $F = 0$ and $G = 0$ can be mapped conformally onto the annulus $1 \leqq |\mu^*_1 + i\mu^*_2| \leqq R$ for some R[7]. Assume that this has been done, that the body maps onto the unit circle, the shock onto the other boundary. Hereafter replace μ^*_α by μ_α. Now proceed to the boundary conditions for conical flow. First observe that to assure that (4.3) and (4.5) will imply the original equations, the analytic function $A + iB$ must be forced to vanish. This can be done by imposing

$$A = 0 \qquad (\text{or } B = 0) \tag{4.6}$$

on both boundaries, and

$$B = 0 \qquad (\text{or } A = 0) \tag{4.7}$$

at *one* point. Also on $| \mu_1 + i\mu_2 | = 1$ the equation of the cone

$$f(X_1, X_2) = 0 \tag{4.8}$$

and the condition of tangency

$$(u_\alpha - wX_\alpha)\, \partial f/\partial X_\alpha = 0 \tag{4.9}$$

must be satisfied, where $X_\alpha = x_\alpha/z$ must be expressed in terms of $\partial u_i/\partial \mu_\beta$ by means of (2.13). Let $(0, 0, Q)$ be the undisturbed velocity ahead of the shock. At the (actually unknown) conical shock $g(X) = 0$, the change in velocity must be normal to the shock, i.e.

$$(\partial g/\partial X_\alpha)/u_\alpha = (X_\beta\, \partial g/\partial X_\beta)(Q - w) \qquad (\alpha \textit{ not } \text{summed}). \tag{4.10}$$

These equations imply $\partial g/\partial X_\alpha = Ku_\alpha$ for some function K, and also

$$Q = X_\alpha u_\alpha + w. \tag{4.11}$$

Then by (2.7), $\Phi = Q$, i.e. *the shock wave is an equipotential surface.* Conversely, any equipotential surface $\Phi = Q =$ constant satisfies (4.10). Finally, on $g = 0$ the equation of the shock polar

$$u_\alpha u_\alpha[\gamma - 1 + 2Q^2 - (\gamma + 1)Qw] = (Q - w)^2[(\gamma + 1)Qw - \gamma + 1] \tag{4.12}$$

must be satisfied, where X_α must be determined as above.

If the constant Q is prescribed, then R has to be found by first determining the flow. It is more convenient to prescribe R and to impose

$$R\, dQ/ds = (-\mu_2\, \partial Q/\partial \mu_1 + \mu_1\, \partial Q/\partial \mu_2) = 0, \tag{4.13}$$

on $| \mu_1 + i\mu_2 | = R$, where Q is defined by (4.11), and where the constant value of Q is to be found after the flow has been determined. The shock wave must likewise be found by (2.13) on this circle after a solution has been found. This is comparable to the situation in numerical computation of Taylor-Maccoll flow, where for prescribed cone angle and surface velocity the free stream velocity and shock angle are not known until the end of the computation.

To recapitulate: The problem is to find on the annulus $1 \leqq | \mu_1 + i\mu_2 | \leqq R$ for given R and $f(X)$ a solution of the elliptic system (4.3), (4.5) which satisfies (4.6), (4.8), and (4.9) on $| \mu_1 + i\mu_2 | = 1$; (4.6), (4.12), and (4.13) on $| \mu_1 + i\mu_2 | = R$; and $B = 0$ (or $A = 0$) at one point. Note that although the free stream's speed is unknown, its direction is fixed. Hence (4.8) determines both the shape of the body and its orientation relative to the undisturbed flow. It should be stressed that mere solution of this boundary value problem is not enough to guarantee the existence of irrotational conical flow about a given cone. In the units employed here, $0 \leqq q \leqq 1$; also Q must be supersonic. Thus a solution is not acceptable unless $(\gamma - 1)^{1/2}(\gamma + 1)^{-1/2} < Q < 1$. Furthermore, the map (2.13) from the hodograph to the physical space must yield a single valued velocity field.

5. Application to Taylor-Maccoll flow. To find the axisymmetric flow about the cone

$$f = (X_\alpha X_\alpha)^{1/2} - \tan\theta_0 = 0, \tag{5.1}$$

of semi-vertex angle θ_0, first seek a solution of the form

$$u_1 = U(t)\cos\phi, \qquad u_2 = U(t)\sin\phi, \qquad w = W(t), \tag{5.2}$$

$$t = (X_\alpha X_\alpha)^{1/2}, \qquad \tan\phi = X_2/X_1\,. \tag{5.3}$$

For the non-isothermic parameters $\mu_1 = t$, $\mu_2 = \phi$, the definitions (2.11) and (2.13) yield

$$q^2 = U^2 + W^2, \qquad g_{11} = U'^2 + W'^2, \qquad g_{12} = 0, \qquad g_{22} = U^2,$$

$$b_{11} = (U'W'' - U''W')(U'^2 + W'^2)^{-1/2}, \qquad b_{12} = 0, \qquad b_{22} = UW'(U'^2 + W'^2)^{-1/2},$$

$$x_1 = -rW'\cos\phi, \qquad x_2 = -rW'\sin\phi, \qquad z = rU'. \tag{5.4}$$

By (5.3) and (5.4)

$$W' + tU' = 0. \tag{5.5}$$

By (2.12)

$$W' = a^2U/[a^2(1 + t^2) - (U - tW)^2], \tag{5.6}$$

where W'' has been eliminated by means of (5.5). At the shock (4.11) and (4.13) yield

$$Q(t) = tU + W, \tag{5.7}$$

$$H(t) \equiv U^2\{2Q^2 - (1 + t^2)[(\gamma + 1)QW - \gamma + 1]\} = 0. \tag{5.8}$$

At the cone $t = \tan\theta_0 = t_0$

$$U_0 = W_0 t_0\,, \tag{5.9}$$

for $U(t_0) = U_0$, $W(t_0) = W_0$.

A short digression on the boundary value problem (5.5) to (5.9) will be instructive. The problem can be replaced by an initial value problem by choosing $W_0 > 0$ such that $W_0^2(1 + t_0^2) < 1$, defining $U_0 = W_0 t_0$, and integrating (5.5) and (5.6). It can be shown that $-U(t)$, $U^2 + W^2$, $Q(t)$, and $-H(t)/U^2(t)$ are increasing functions of t, that $H(t_0) > 0$, and that $H = 0$ before a^2U or a^2U/W' can vanish. By (5.7) and (5.8)

$$0 < Ut = 2t^2[Q^2 - (1 + t^{-2})A^2]/(\gamma + 1)(1 + t^2)Q$$

where $A^2 = 1/2(\gamma - 1)(1 - Q^2)$. Hence Q is *supersonic*. If $W_0(1 + t_0^2) \geqq 1$, $Q(t) > 1$ at $H = 0$. Thus some solutions of (5.5) and (5.6) are rejected by the restriction $Q < 1$. It is also known that there exists $q^*(t_0)$ such that for $W_0(1 + t_0^2)^{1/2} \leqq q^*(t_0)$, $(U^2 + W^2)^{1/2}$ is subsonic all the way up to the shock; for $q^*(t_0) < W_0(1 + t_0^2)^{1/2} \leqq (\gamma - 1)^{1/2}(\gamma + 1)^{-1/2}$ the flow is subsonic near the cone, supersonic near the shock; and for $(\gamma - 1)^{1/2}(\gamma + 1)^{-1/2} < W_0(1 + t_0^2)^{1/2} < 1$ the flow is completely supersonic. To obtain completely supersonic flow and $Q < 1$ one must have $t_0^2 < 2/(\gamma - 1)$. For $\gamma = 1.4$ this yields the crude estimate $\theta_0 < 66°$, by contrast with the computed bound of 57.5°. A similar crude bound for finite free stream Mach numbers can be obtained by replacing the upper bound 1 for Q by $M[M^2 + 2/(\gamma - 1)]^{-1/2}$. In general, for a given (small enough) t_0 there is a minimum free stream Mach number for Taylor-Maccoll flow. For greater Mach numbers and the same t_0 there are two solutions, one of which has a strong, the other a weak shock.

Isothermic parameters $\mu_1^* = T(t)$, $\mu_2^* = \phi$ can be defined by

$$[a^2(U'^2 + W'^2) - (UU' + WW')^2](dt/dT)^2 = a^2U^2.$$

By (5.5) and (5.6) this becomes $(dT/dt)^2 = W'/Ut^2$. Then choose

$$T(t) = 1 + \int_{t_0}^{t} (W'/Ut^2)^{1/2}\, dt.$$

At the shock let $t = \tan \theta_w = t_w$, and $T(t_w) = R$.

$$R = 1 + \int_{t_0}^{t_w} (W'/Ut^2)^{1/2}\, dt. \tag{5.10}$$

In polar coordinates T, ϕ, the cone corresponds to $T = 1$, the shock to $T = R$. Approximate values for R as functions of free stream Mach number and cone semi-angle for $\gamma = 1.4$, computed on the ENIAC and ORDVAC, are shown in Fig. 1. The curve for $\theta = 40°$ (not shown) practically coincides with that for $\theta = 35°$ above $M = 3$.

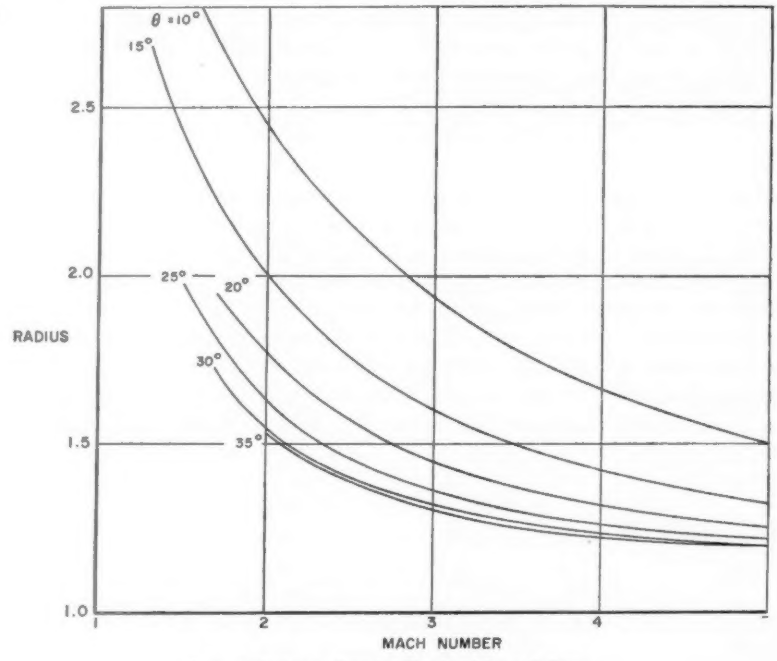

FIG. 1. Outer radius for Taylor-Maccoll flow.

6. Generalities and speculations. A. Numerical solutions of the boundary value problem can be attempted by relaxation methods. Estimates of the outer radius, R, of the annulus for prescribed free stream Mach number can be based on Fig. 1. The treatment of boundary conditions and the choice of coordinate lattice will be greatly simplified by the transformation $\mu_1^* + i\mu_2^* = \log(\mu_1 + i\mu_2)$ which maps the annulus $1 \leqq |\mu_1 + i\mu_2| \leqq R$ onto the rectangle $0 \leqq \mu_1^* \leqq \log R$, $0 \leqq \mu_2^* \leqq 2\pi$. Computation

could be directed toward a great variety of goals, such as (i) comparisons with Kopal's yawing cone calculations, based on A. H. Stone's theory [20]; (ii) checks of Moeckel's three-dimensional characteristic method for calculating the same types of flows [15]; (iii) use of computed results to estimate the importance of Ferri's singularity in the entropy [8]; (iv) determinations of limitations on shape, such as (a) maximum inclination between a surface element and the incident flow; (b) maximum eccentricity for elliptic cones, i.e., can one pass to the limiting case of the delta wing at an angle of attack; (c) maximum yaw for a circular cone, i.e. can the angle of attack exceed the semicone angle?

B. Since Taylor-Maccoll flow is a solution of this type of boundary value problem, it might be possible to apply the iterative method described in [6, p. 287], Green's function for an annulus being known, to show that flows can be constructed about slightly yawing almost circular cones $0 = f(X_1, X_2) = X_\alpha X_\alpha - \tan^2 \theta_0 + \epsilon g(X_1, X_2)$ for reasonable g (not a function of $X_\alpha X_\alpha$ only) and small enough ϵ.

C. M. Shiffman [17] has recently developed a proof of the existence and uniqueness of subsonic plane flows about fairly general shapes. This involves reformulating of the plane flow problem in variational form and then applying direct methods in the calculus of variations. It can be shown that this is impossible for the system (4.3), (4.5), i.e. that these are not the Euler equations for any integral

$$I = \iint F(\mu_1, \mu_2, u, \partial u/\partial \mu_1, \partial u/\partial \mu_2)\, d\mu_1\, d\mu_2 . \tag{6.1}$$

D. Can a new linearization of the conical flow equations be based on (4.3), e.g. by suppressing the right members? Instead of defining in advance the transformation from the X_1X_2-plane to the $\mu_1\mu_2$-plane, as is done in the usual linearization, use (2.13) for this purpose. If possible, retain all of the boundary conditions in Sec. 4.

7. Remarks on plane flow. Shiffman [17] and Bers [unpublished results] have recently established existence and uniqueness theorems for completely subsonic flows about plane profiles. Accordingly, it may be significant that their boundary value problem can be restated, as shown below, in a form intimately related to the problem of Sec. 4.

Since $u_\alpha(x_1, x_2)$ are functionally independent in subsonic plane flow, apply the Legendre transformation

$$k(u_1, u_2) = \phi - x_\alpha u_\alpha , \tag{7.1}$$

to (2.1) to obtain

$$(a^2\delta_{\alpha\beta} - u_\alpha u_\beta)(-1)^{\alpha+\beta}\, \partial^2 k/\partial u_{\alpha+1}\, \partial u_{\beta+1} = 0, \tag{7.2}$$

where $\alpha + 1$ and $\beta + 1$ are to be reduced mod 2, and where the velocity u_α is assigned to the point

$$x_\alpha = -\partial k/\partial u_\alpha . \tag{7.3}$$

Now introduce the more general parameters $\mu_\alpha = \mu_\alpha(u_1, u_2)$. Then

$$u_\alpha = u_\alpha(\mu_1, \mu_2). \tag{7.4}$$

Let

$$Q^2 = u_\alpha u_\alpha , \qquad Q\, \partial Q/\partial \mu_\beta = u_\alpha\, \partial u_\alpha/\partial \mu_\beta , \qquad G_{\alpha\beta} = (\partial u_\gamma/\partial \mu_\alpha)(\partial u_\gamma/\partial \mu_\beta) \tag{7.5}$$

and let the third order determinants

$$\det \begin{vmatrix} k_{,\alpha\beta} & \partial k/\partial \mu_\delta \\ u_{\gamma,\alpha\beta} & \partial u_\gamma/\partial \mu_\delta \end{vmatrix} = B_{\alpha\beta}\,, \tag{7.6}$$

where $k_{,\alpha\beta}$ and $u_{\gamma,\alpha\beta}$ are the second covariant derivatives of the scalars k and u_γ with respect to μ_α and μ_β based on $G_{\alpha\beta}$. For the special choice $\mu_\alpha = u_\alpha$ (7.2) implies

$$(a^2 G_{\alpha\beta} - Q^2\,\partial Q/\partial\mu_\alpha\,\partial Q/\partial\mu_\beta)(-1)^{\alpha+\beta}B_{\alpha+1\,\beta+1} = 0. \tag{7.7}$$

Since $G_{\alpha\beta}$ and $\partial Q/\partial\mu_\beta$ are tensors, while $B_{\alpha\beta}$ is a relative tensor of weight one, the form of (7.7) is independent of the particular choice of parameters μ_α. Also observe that (7.3) implies

$$x_\alpha\,\partial u_\alpha/\partial\mu_\beta + \partial k/\partial\mu_\beta = 0. \tag{7.8}$$

For subsonic flow choose *isothermic parameters* defined by

$$A \equiv a^2 u_{\alpha 1}u_{\alpha 2} - u_\alpha u_{\alpha 1}u_\beta u_{\beta 2} = 0, \tag{7.9}$$

$$2B \equiv a^2(u_{\alpha 1}u_{\alpha 1} - u_{\alpha 2}u_{\alpha 2}) - (u_\alpha u_{\alpha 1})^2 + (u_\alpha u_{\alpha 2})^2 = 0. \tag{7.10}$$

Now by (7.6) and (7.7)

$$\partial^2 k/\partial\mu_\alpha\,\partial\mu_\alpha = D_\beta\,\partial k/\partial\mu_\beta\,, \qquad \partial^2 u_\gamma/\partial\mu_\alpha\,\partial\mu_\alpha = D_\beta\,\partial u_\gamma/\partial\mu_\beta\,. \tag{7.11}$$

Since (7.5), (7.9), and (7.10) can be obtained by setting $w \equiv 0$ in (2.11), (4.1), and (4.2), then by analogy with (4.5)

$$(a^2 G_{\alpha\beta} - Q^2\,\partial Q/\partial\mu_\alpha\,\partial Q/\partial\mu_\beta)D_\beta = 0.5G_{\beta\beta}\,\partial(a^2 + Q^2)/\partial\mu_\alpha - G_{\alpha\beta}\,\partial a^2/\partial\mu_\beta\,. \tag{7.12}$$

In the present case D_β are rational functions of the six arguments u_α, $\partial u_\alpha/\partial\mu_\beta$. As in Section 4, if $k(\mu)$ and $u_\gamma(\mu)$ are solutions of (7.11) where D_β are defined by (7.12), then $A + iB$ is an analytic function of $\mu_1 + i\mu_2$.

Suppose that the airfoil is described by $f(x_1, x_2) = 0$. Assume that $f = 0$ is transformed into a closed curve $F(\mu_1, \mu_2) = 0$, and that the flow is mapped onto the exterior of $F = 0$. Since (7.9) and (7.10) are invariant under conformal transformation, it may be assumed with no loss of generality that $F = 0$ is $|\mu_1 + i\mu_2| = 1$, and that the point at infinity in the $x_1 + ix_2$ plane maps onto the point at infinity in the $\mu_1 + i\mu_2$ plane. On the unit circle impose the boundary conditions A (or B) $= 0$; $f(x_1, x_2) = 0$, where x_α are defined by (7.8); and $u_\alpha \partial f/\partial x_\alpha = 0$. At infinity $u_\alpha = U_\alpha =$ constants; and $x_1 + ix_2 = \infty$. Finally, at some point impose B (or A) $= 0$.

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FREE VIBRATION OF A RECTANGULAR PLATE WITH DAMPING CONSIDERED*

BY

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Introduction. This paper presents a method for calculating the natural frequencies of the normal modes of free vibration of a rectangular plate, fixed along each edge, with arbitrary shape ratio. Viscous damping of the plate material is considered.

Generally speaking, the objects of this paper are (a) to present a simple and practical solution for the stated problem, (b) to develop this solution in closed form so that a designer can evaluate the natural frequencies of a clamped plate of any shape ratio in a relatively short time, and (c) to make possible the determination of the damping coefficient by comparing experimental and theoretical values.

The theory developed is subjected to the following restrictions: (1) the plate is composed of a material which follows Hooke's law; (2) the deflection of the plate is small compared to its thickness; (3) the thickness of the plate is small compared to its lateral dimensions. The method which is used to solve this problem is that of Galerkin[1] [1] and belongs to the same general class as those of Rayleigh and Ritz. This method can be used (a) to determine an approximate solution of a differential equation with given boundary conditions admitting only the functions which satisfy the prescribed boundary condition exactly, and (b) to treat the problems which belong to non-conservative systems. This method was described by E. P. Grossman [2] and also by W. J. Duncan [3], [4].

The degree of accuracy expected can be increased by increasing the number of independent functions which are used in the solution.

W. J. Duncan has shown in his papers that when the functions are well chosen an excellent approximation can be obtained by use of a very small number of admissible functions. Since our system is non-conservative this method is most convenient for the solution of the problem.

Solution of the problem. If damping forces are proportional to velocity, the motion of the plate is governed by the following partial differential equation:

$$\nabla^4 w(x, y, t) + \frac{\rho h}{D} w_{tt}(x, y, t) = -\frac{k}{D} w_t(x, y, t), \tag{1}$$

where $w(x, y, t)$ is the transverse deflection of the plate, k is the damping coefficient, h is thickness of the plate, ρ is the mass density of the plate material and $D = Eh^3/12(1 - \nu^2)$ is the flexural rigidity of the plate. For a uniform plate vibrating harmonically with an amplitude $\phi(x, y)$ we can write

$$w(x, y, t) = \phi(x, y) \exp(-\alpha t) \cos \omega t, \tag{2}$$

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[1]Numbers in square brackets refer to the Bibliography at the end of the paper.

where ω is the angular frequency. Equations (1) and (2) lead to

$$\nabla^4\phi(x, y) \cos \omega t + \phi(x, y)\left\{\left[\frac{\rho h}{D}(\alpha^2 - \omega^2) - \alpha\frac{k}{D}\right] \cos \omega t + \left(2\frac{\rho h}{D}\alpha\omega - \omega\frac{k}{D}\right)\sin \omega t\right\} = 0. \quad (3)$$

Since Eq. (3) must be satisfied for all values of t we obtain $(w/D)\ (2\rho h\alpha - k) = 0$. Therefore

$$\alpha = \frac{k}{2\rho h}. \quad (4)$$

Now, Eq. (3) becomes

$$\nabla^4\phi(x, y) - \left[\alpha\frac{k}{D} - \frac{\rho h}{D}(\alpha^2 - \omega^2)\right]\phi(x, y) = 0$$

or

$$\nabla^4\phi(x, y) - \lambda\phi(x, y) = 0, \quad (5)$$

where

$$\lambda = \frac{k^2}{4\rho h D}\left[1 + \left(\frac{2\rho h\omega}{k}\right)^2\right]. \quad (6)$$

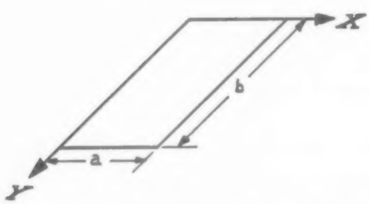

FIG. 1. Coordinate System for the Plate

Assume the solution of Eq. (5) to be given in the form

$$\phi(x, y) = \sum_{r=1}^{m}\sum_{s=1}^{n} a_{rs}X_rY_s \qquad (r, s = 1, 2, 3, \cdots), \quad (7)$$

in which the X_r and Y_s are admissible functions that satisfy only geometrical boundary conditions, but need not satisfy any "natural boundary conditions", and a_{rs} are the amplitude coefficients. Since the function $\phi(x, y)$ is an approximate solution of the problem, then the error caused in the solution is of the magnitude

$$\epsilon = \sum_{r=1}^{m}\sum_{s=1}^{n} a_{rs}[\nabla^4(X_rY_s) - \lambda X_rY_s]. \quad (8)$$

The criterion that the approximation is best if ϵ tends to zero requires, as stated by Galerkin, that the integrals

$$J = \left|\int_0^a\int_0^b \epsilon X_pY_q\,dx\,dy\right| \qquad (p, q = 1, 2, 3, \cdots) \quad (9)$$

be a minimum. The lateral dimensions of the plate are a and b. Equations (8) and (9) lead to

$$\sum_{r=1}^{m} \sum_{s=1}^{n} a_{rs} \int_0^a \int_0^b [\nabla^4(X_r Y_s) - \lambda X_r Y_s] X_p Y_q \, dx \, dy = 0 \tag{10}$$

which can be written in the form

$$\sum_{r=1}^{m} \sum_{s=1}^{n} a_{rs} \int_0^a \int_0^b [X_{r,xxxx} Y_s + 2X_{r,xx} Y_{s,yy} + X_r Y_{s,yyyy} - \lambda X_r Y_s] X_p Y_q \, dx \, dy = 0. \tag{11}$$

Equation (11) represents a system of linear homogeneous equations in the unknown coefficients a_{rs}. The natural frequencies $\lambda_1, \lambda_2, \cdots$ are determined from the condition that the determinant of the system must vanish.

Let the appropriate characteristic functions X_r, Y_s be given, following Young [5], in the form

$$\Psi_v(\xi) = \left(\cosh \mu_v \frac{\xi}{l} - \cos \mu_v \frac{\xi}{l}\right) - \beta_v\left(\sinh \mu_v \frac{\xi}{l} - \sin \mu_v \frac{\xi}{l}\right), \tag{12}$$

such that

$$\text{if} \quad v = r, \quad \Psi_v = X_r, \quad \xi = x, \quad l = a;$$

$$\text{if} \quad v = s, \quad \Psi_v = Y_s, \quad \xi = y, \quad l = b.$$

The function $\Psi_v(\xi)$ has to satisfy the following boundary conditions:

$$\Psi_v(0) = \Psi_v(l) = \Psi_{v,\xi}(0) = \Psi_{v,\xi}(l) = 0. \tag{13}$$

Then

$$\cosh \mu_v \cos \mu_v - 1 = 0, \tag{14}$$

$$\beta_v = \frac{\cosh \mu_v - \cos \mu_v}{\sinh \mu_v - \sin \mu_v}. \tag{15}$$

The values of β_v and μ_v are presented in Table I. Since the characteristic functions are

TABLE I

Values of β_v and μ_v

v	β_v	μ_v	μ_v^4
1	0.98250	4.73004	500.564
2	1.00078	7.85320	3,803.537
3	0.99997	10.99561	14,617.630
4	1.00000	14.13717	39,943.799
5	1.00000	17.27876	89,135.407
6	1.00000	20.42035	173,881.316
$v > 6$	1.00000	$(2v + 1)\,\pi/2$	$(2v + 1)^4\,\pi^4/16$

of the form of Eq. (12), we obtain

$$X_{r,xxxx} = \frac{\mu_r^4}{a^4} X_r , \qquad Y_{s,yyyy} = \frac{\mu_s^4}{b^4} Y_s , \tag{16}$$

$$\int_0^a X_p X_r \, dx = a\delta_p^r , \text{ where } \delta_p^r = \begin{cases} 1 & \text{if} \quad p = r \\ 0 & \text{if} \quad p \neq r \end{cases} \tag{17}$$

and

$$\int_0^b Y_q Y_s \, dy = b\delta_q^s , \text{ where } \delta_q^s = \begin{cases} 1 & \text{if} \quad q = s \\ 0 & \text{if} \quad q \neq s. \end{cases} \tag{18}$$

Using the abbreviation

$$H_{pr} = a \int_0^a X_p X_{r,xx} \, dx, \qquad H_{qs} = b \int_0^b Y_q Y_{s,yy} \, dy,$$

knowing that

$$\int_0^l \Psi_i \Psi_{j,\xi\xi} \, d\xi = | \Psi_i \Psi_{j,\xi} |_0^l - \int_0^l \Psi_{i,\xi} \Psi_{j,\xi} \, d\xi$$

and using boundary condition Eq. (13), we have

$$\int_0^l \Psi_i \Psi_{j,\xi\xi} \, d\xi = -\int_0^l \Psi_{i,\xi} \Psi_{j,\xi} \, d\xi. \tag{19}$$

TABLE II

Values of $l \int_0^l \psi_{i,\xi} \Psi_{j,\xi} \, d\xi$

i \ j	1	2	3	4	5	6
1	12.30262	0	−9.73079	0	−7.61544	0
2	0	46.05012	0	−17.12892	0	−15.19457
3	−9.73079	0	98.90480	0	−24.34987	0
4	0	−17.12892	0	171.58566	0	−31.27645
5	−7.61544	0	−24.34987	0	263.99798	0
6	0	−15.19457	0	−31.27645	0	376.15008

Numerical values for the integrals, Eq. (19), are presented in Table II. Hence

$$H_{pr} = -a \int_0^a X_{p,x} X_{r,x} \, dx, \tag{20}$$

$$H_{qs} = -b \int_0^b Y_{q,y} Y_{s,y} \, dy. \tag{21}$$

Now Eq. (11) becomes

$$\sum_{r=1}^{m} \sum_{s=1}^{n} a_{rs} \left\{ \int_0^a \int_0^b [X_p X_{r,xxxx} Y_q Y_s + X_p X_r Y_q Y_{s,yyyy} \right.$$
$$\left. + 2X_p X_{r,xx} Y_q Y_{s,yy}] \, dx \, dy - \lambda \int_0^a \int_0^b X_r X_p Y_q Y_s \, dx \, dy \right\} = 0$$

or

$$\sum_{r=1}^{m} \sum_{s=1}^{n} a_{rs} \left\{ \int_0^a \frac{\mu_r^4}{a^4} X_p X_r \, dx \int_0^b Y_q Y_s \, dy + \int_0^a X_p X_r \, dx \int_0^b \frac{\mu_s^4}{b^4} Y_q Y_s \, dy \right.$$
$$\left. + 2 \int_0^a X_p X_{r,xx} \, dx \int_0^b Y_q Y_{s,yy} \, dy - \lambda \int_0^a X_p X_r \, dx \int_0^b Y_q Y_s \, dy \right\} = 0. \tag{22}$$

After substitution of the corresponding values, and multiplication of each term by a^2 we obtain

$$\sum_{r=1}^{m} \sum_{s=1}^{n} a_{rs} \left\{ \left[\left(\frac{\mu_r^4}{a^4} + \frac{\mu_s^4}{b^4} \right) a^3 b \delta_{pq}^{(rs)} + 2 \frac{a^2}{ab} H_{pr} H_{qs} \right] - \lambda a^3 b \delta_{pq}^{(rs)} \right\} = 0, \tag{23}$$

where

$$\delta_{pq}^{(rs)} = \int_0^a X_p X_r \, dx \int_0^b Y_q Y_s \, dy. \tag{24}$$

Equation (23) may be rewritten as

$$\sum_{r=1}^{m} \sum_{s=1}^{n} a_{rs} \left\{ \left[\left(\frac{b}{a} \mu_r^4 + \frac{a^3}{b^3} \mu_s^4 \right) \delta_{pq}^{(rs)} + 2 \frac{a}{b} H_{pr} H_{qs} \right] - \Lambda \delta_{pq}^{(rs)} \right\} = 0, \tag{25}$$

in which

$$\Lambda = \lambda a^3 b = \frac{a^3 b k^2}{4\rho h D} \left[1 + \left(\frac{2\rho h \omega}{k} \right)^2 \right]. \tag{26}$$

Let

$$E_{pq}^{(rs)} = \left[\left(\frac{b}{a} \right) \mu_r^4 + \left(\frac{a}{b} \right)^3 \mu_s^4 \right] \delta_{pq}^{(rs)} + 2 \frac{a}{b} H_{pr} H_{qs} \, . \tag{27}$$

Then Eq. (25) becomes

$$\sum_{r=1}^{m} \sum_{s=1}^{n} a_{rs} [E_{pq}^{(rs)} - \Lambda \delta_{pq}^{(rs)}] = 0, \tag{28}$$

where

$$\delta_{pq}^{(rs)} = \begin{cases} 1 & \text{for} \quad p = r \quad \text{and} \quad q = s \\ 0 & \text{if either} \quad p \neq r \quad \text{or} \quad q \neq s \end{cases}$$

and

$$\begin{aligned} E_{pq}^{(rs)} &= \frac{b}{a} \mu_r^4 + \left(\frac{a}{b} \right)^3 \mu_s^4 + 2 \frac{a}{b} H_{rr} H_{ss} \quad \text{for} \quad p = r \quad \text{and} \quad q = s, \\ E_{pq}^{(rs)} &= 2 \frac{a}{b} H_{pr} H_{qs} \quad \text{if either} \quad p \neq r \quad \text{or} \quad q \neq s. \end{aligned} \tag{29}$$

Equation (28) is a system of mn linear homogeneous equations in mn unknowns a_{rs} ($r = 1, 2, \cdots m$; $s = 1, 2, \cdots n$). Since a_{rs} cannot all be zero, the determinant of the system must be zero. This leads, in general, to an algebraic equation of degree mn in Λ.

From Eq. (26) it follows that

$$\omega = \left[\frac{\Lambda D}{\rho h a^3 b} - \left(\frac{k}{2\rho h}\right)^2\right]^{1/2}. \tag{30}$$

Then for any shape ratio $\sigma = b/a$, one obtains

$$\omega = \left[\frac{\Lambda D}{\rho h \sigma a^4} - \left(\frac{k}{2\rho h}\right)^2\right]^{1/2}. \tag{31}$$

If $k = 0$, then

$$\Lambda = \frac{\omega^2 \rho h \sigma a^4}{D} \tag{32}$$

which is the eigenvalue parameter for the plate with fixed boundary conditions. From Eq. (30) it can be seen that the frequency decreases with an increase in the value of the damping factor k. The zero value of ω is obtained for

$$k = \frac{2}{a^2}\left(\rho h D \frac{\Lambda}{\sigma}\right)^{1/2}. \tag{33}$$

The frequencies are calculated for the first three modes of a square plate fixed along each edge, and the results are presented in Table IV.

TABLE III

Coefficients for Vibration of Damped Square Plate

r	s	$E^{(rs)}_{11}$	$E^{(rs)}_{12}$	$E^{(rs)}_{13}$	$E^{(rs)}_{21}$	$E^{(rs)}_{22}$	$E^{(rs)}_{23}$	$E^{(rs)}_{31}$	$E^{(rs)}_{32}$	$E^{(rs)}_{33}$
1	1	1,303.84	0	−239.43	0	0	0	−239.43	0	189.38
1	2	0	5,437.18	0	0	0	0	0	−896.21	0
1	3	−239.43	0	17,551.77	0	0	0	189.38	0	−1,924.84
2	1	0	0	0	5,437.18	0	−896.21	0	0	0
2	2	0	0	0	0	11,848.30	0	0	0	0
2	3	0	0	0	−896.21	0	27,530.32	0	0	0
3	1	−239.43	0	189.38	0	0	0	17,551.77	0	−1,924.84
3	2	0	−896.21	0	0	0	0	0	27,530.32	0
3	3	189.38	0	−1,924.84	0	0	0	−1,924.84	0	48,799.58

TABLE IV

The Values of $\Lambda^{1/2}$ of Vibration of Square Plate Fixed Along Each Edge

$$\omega = \left[\frac{\Lambda D}{\rho h \sigma a^4} - \left(\frac{k}{2\rho h}\right)^2\right]^{1/2}$$

Mode	1st	2nd	3rd
$\Lambda^{1/2}$	36.11	73.73	108.85
(D. Young)	(35.99)	(73.41)	(108.27)

Table III represents the coefficients $E_{pq}^{(rs)}$ for the vibration of a clamped square plate. Using only the first two terms of the series for $E_{pq}^{(rs)}$, we obtain approximations for

$\Lambda^{1/2}$ which are compared with the D. Young solution [5] (given in parenthesis in Table IV). It can be seen that a good agreement between the results is obtained.

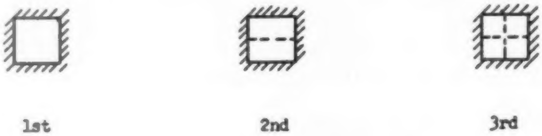

Fig. 2. Nodal Line for Square Plate

Conclusion. The problem of free vibrations of a rectangular plate fixed along each edge and having internal viscous damping is presented and solved by means of generalized Galerkin's method. By choosing the function $\Psi(\xi)$ to satisfy the boundary conditions, Galerkin's method can be applied to a plate with any type of support, even when damping forces are present.

The influence of the damping factor on the natural frequency is given by Eq. (30). It can be seen that the natural frequency decreases with increase in the k-values.

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ON THE DEFORMATION OF ELASTIC SHELLS OF REVOLUTION*

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1. Introduction. In a recent paper, E. Reissner [1] formulated a theory for finite deformation of elastic isotropic shells of revolution where the theory of small deformation (linear theory) is also discussed. In the present note, there is derived a single complex differential equation for small deformation of shells of revolution which is valid for uniform thickness, as well as for a large class of variable thickness.

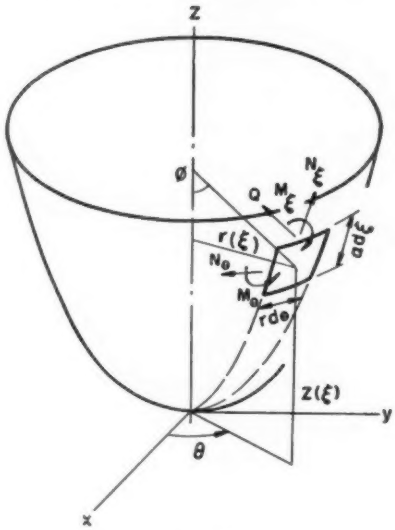

FIG. 1

With the use of cylindrical coordinates r, θ, z, the parametric equation of the middle surface of the shell (see Fig. 1) may be represented by

$$r = r(\xi), \qquad z = z(\xi). \tag{1.1}$$

Denoting by ϕ the inclination of the tangent to the meridian of the shell, then

$$r' = \alpha \cos \phi, \qquad z' = \alpha \sin \phi, \tag{1.2}$$

where

$$\alpha = [(r')^2 + (z')^2]^{1/2} \tag{1.3}$$

and prime denotes differentiation with respect to ξ.

We note for future reference that the principal radii of curvature r_1 and r_2 are, re-

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spectively, the radius of the curvature of the curve generating the middle surface and the length of the normal intercepted between this curve (generating curve) and the axis of rotation. It follows from the geometry of the middle surface that

$$r = r_2 \sin \phi. \tag{1.4}$$

The stress resultants N_ξ, N_θ and Q, and the stress couples M_ξ and M_θ, acting on an element of the shell, are shown in Fig. 1. Also, as in [1], it is convenient to introduce "horizontal" and "vertical" stress resultants, H and V, given by

$$\alpha N_\xi = r'H + z'V, \qquad \alpha Q = -z'H + r'V. \tag{1.5}$$

We now record the basic equations of the small deflection theory of elastic shells of revolution with axisymmetric loading, as given by Reissner in [1].

$$\begin{aligned}
rV &= -\int r\alpha p_v \, d\xi, \\
\alpha N_\theta &= (rH)' + r\alpha p_H, \\
rN_\xi &= (rH)\cos\phi + (rV)\sin\phi, \\
rQ &= -(rH)\sin\phi + (rV)\cos\phi, \\
M_\xi &= \frac{D}{\alpha}\left[\beta' + \nu\frac{r'}{r}\beta\right], \\
M_\theta &= \frac{D}{\alpha}\left[\frac{r'}{r}\beta + \nu\beta'\right], \\
u &= \frac{r}{Eh}(N_\theta - \nu N_\xi), \\
w &= \int\left[\frac{z'}{C}(N_\xi - \nu N_\theta) - r'\beta\right] d\xi,
\end{aligned} \tag{1.6}$$

where β is the negative change in ϕ due to deformation; u and w are the components of displacement in the radial and axial directions; p_H and p_V denote the components of load intensity in r and z directions; h is the thickness of the shell; and

$$C = Eh, \qquad D = \frac{Eh^3}{12(1-\nu^2)}, \tag{1.7}$$

E and ν being Young's modulus and Poisson's ratio, respectively.

With β and rH as basic variables, proper elimination between Eqs. (1.6), differential equations of equilibrium and compatibility, leads to the following two second-order differential equations:

$$\beta'' + \frac{(rD/\alpha)'}{(rD/\alpha)}\beta' - \left[\left(\frac{r'}{r}\right)^2 - \nu\frac{(r'D/\alpha)'}{(rD/\alpha)}\right]\beta + \frac{z'}{(rD/\alpha)}(rH) = \frac{r'}{(rD/\alpha)}(rV), \tag{1.8}$$

$$\begin{aligned}
(rH)'' + \frac{(r/\alpha C)'}{(r/\alpha C)}(rH)' - \left[\left(\frac{r'}{r}\right)^2 + \nu\frac{(r'/\alpha C)'}{(r/\alpha C)}\right](rH) \\
- \frac{z'}{(r/\alpha C)}\beta = \left[\frac{r'z'}{r^2} + \nu\frac{(z'/\alpha C)'}{(r/\alpha C)}\right](rV) + \nu\frac{z'}{r}(rV)' \\
- \left[\frac{(r/\alpha C)'}{(r/\alpha C)} + \nu\frac{r'}{r}\right](r\alpha p_H) - (r\alpha p_H)'.
\end{aligned} \tag{1.9}$$

2. Normal form of the differential equations. Substitution of the quantities C and D from (1.7) into (1.8) and (1.9) and rearrangement of terms result in

$$L_1(\beta) + \nu\left[\frac{(r'/\alpha)'}{(r/\alpha)} + 3\,\frac{(r'h')}{(rh)}\right]\beta + \frac{\alpha^2 m}{r_2 h_0}\left(\frac{h_0}{h}\right)\psi = \frac{\alpha^2 m}{r_2 h_0}\left(\frac{h_0}{h}\right)\left(\frac{mrV}{Eh^2}\right)\cot\phi, \tag{2.1}$$

$$\begin{aligned} L_1(\psi) - \nu\left[\frac{(r'/\alpha)'}{(r/\alpha)} + 3\,\frac{(r'h')}{(rh)}\right]\psi - \frac{\alpha^2 m}{r_2 h_0}\left(\frac{h_0}{h}\right)\beta \\ + 2\left[\frac{h''}{h} + 2\nu\,\frac{(r'h')}{(rh)} + \frac{(r/\alpha)'}{(r/\alpha)}\,\frac{h'}{h}\right]\psi = Z\,\frac{m}{Eh^2}, \end{aligned} \tag{2.2}$$

where Z denotes the right-hand side of (1.9), h_0 is the value of h at some reference section (say $\xi = \xi_0$), and

$$\begin{aligned} L_1(\ \ ) &\equiv (\ \ )'' + \left[\frac{(r/\alpha)'}{(r/\alpha)} + 3\,\frac{h'}{h}\right](\ \ )' - \left(\frac{r'}{r}\right)^2(\ \ ), \\ \psi &= \frac{mrH}{Eh^2}, \qquad m = [12(1-\nu^2)]^{1/2}. \end{aligned} \tag{2.3}$$

In (2.1) and (2.2), let

$$\frac{\alpha^2 m}{r_2 h_0} = 2\mu^2 f(\xi), \tag{2.4}$$

where μ is constant and it is to be noted that $f(\xi)$ is independent of the thickness $h(\xi)$. Then, multiplication throughout (2.1) and (2.2) by $\{h[h_0 f(\xi)]^{-1}\}$ results in

$$L(\beta) + \nu\lambda\beta + 2\mu^2\psi = F, \tag{2.5}$$

$$L(\psi) - (\nu\lambda - \delta)\psi - 2\mu^2\beta = G, \tag{2.6}$$

where

$$\begin{aligned} L(\ \ ) &\equiv \left[\frac{h_0}{h}\,f(\xi)\right]^{-1} L_1(\ \ ), \\ \lambda &= \left[\frac{h_0}{h}\,f(\xi)\right]^{-1}\left\{\frac{(r'/\alpha)'}{r/\alpha} + \frac{3r'h'}{rh}\right\}, \\ \delta &= 2\left[\frac{h_0}{h}\,f(\xi)\right]^{-1}\left\{\frac{h''}{h} + 2\nu\,\frac{(r'h')}{(rh)} + \frac{(r/\alpha)'}{(r/\alpha)}\,\frac{h'}{h}\right\}, \\ F &= 2\mu^2\,\frac{mrV}{Eh^2}\cot\phi, \\ G &= \frac{m}{Eh^2}\left[\frac{h_0}{h}\,f(\xi)\right]^{-1} Z. \end{aligned} \tag{2.7}$$

Introducing the complex function

$$U = \beta + ik\psi; \qquad i = \sqrt{-1} \tag{2.8}$$

where k, an arbitrary function of ξ is to be determined, the differential equations (2.5) and (2.6) may be combined to read

$$L(U) = 2\mu^2\left(ik - \frac{\nu\lambda}{2\mu^2}\right)\left\{\beta + \frac{\left[ik\left(\frac{\nu\lambda}{2\mu^2} - \frac{\delta}{2\mu^2}\right) - 1\right]}{\left(ik - \frac{\nu\lambda}{2\mu^2}\right)}\psi\right\}$$
$$+ i\left[\frac{h_0}{h} f(\xi)\right]^{-1}\left\{k'' + \left[2\frac{\psi'}{\psi} + \frac{(r/\alpha)'}{(r/\alpha)} + 3\frac{h'}{h}\right]k'\right\}\psi \tag{2.9}$$
$$+ (F + ikG).$$

Taking k in the form

$$k = -i\frac{1}{2\mu^2}\left(\nu\lambda - \frac{\delta}{2}\right) + \left\{1 - \left[\frac{1}{2\mu^2}\left(\nu\lambda - \frac{\delta}{2}\right)\right]^2\right\}^{1/2} \tag{2.10}$$

with the restriction (the implication of this restriction will be discussed later) that

$$k' = k'' = 0, \tag{2.11}$$

(2.9) transforms into

$$L(U) = 2\mu^2\left(ik - \frac{\nu\lambda}{2\mu^2}\right)U + (F + ikG). \tag{2.12}$$

By putting the last complex differential equation in the form

$$L_1(U) - i\frac{2\mu^2 h_0\left(k + i\frac{\nu\lambda}{2\mu^2}\right)}{hf^{-1}(\xi)}U = \frac{h_0}{hf^{-1}(\xi)}(F + ikG)$$

and observing that the coefficient of U', resulting from the application of the operator L_1, defined by (2.3), is

$$R = \left[\frac{(r/\alpha)'}{(r/\alpha)} + 3\frac{h'}{h}\right],$$

and that

$$\exp\left[\frac{1}{2}\int R\, d\xi\right] = h^{3/2}\left(\frac{r}{\alpha}\right)^{1/2}$$

then, with the aid of the transformation

$$W = \left(\frac{h}{h_0}\right)^{3/2}\left(\frac{r}{\alpha}\right)^{1/2}U, \tag{2.13}$$

we obtain

$$W'' + [2i^3\mu^2\Psi^2(\xi) + \Lambda(\xi)]W = \left[\frac{h}{h_0}\frac{r}{\alpha}\right]^{1/2} f(\xi)[F + ikG] \tag{2.14}$$

which is the normal form of (2.12), and where

$$\Psi^2 = \left(k + i\frac{\nu\lambda}{2\mu^2}\right)\left(\frac{h_0}{h}\right)f(\xi),$$
$$\Lambda = -\frac{1}{2}\frac{(r/\alpha)''}{(r/\alpha)} + \frac{1}{4}\left[\frac{(r/\alpha)'}{(r/\alpha)}\right]^2 - \left(\frac{r'}{r}\right)^2 - \frac{3}{2}\frac{(r/\alpha)'}{(r/\alpha)}\frac{h'}{h} - \frac{3}{2}\frac{h''}{h} - \frac{3}{4}\left(\frac{h'}{h}\right)^2. \tag{2.15}$$

We now return to (2.12) and observe that condition (2.11) is fulfilled only if k is a constant, and this may be achieved by proper choice of λ and δ. In particular, we note the following two cases.

(a) For shells of variable thickness and with reference to differential equation (2.12), the condition (2.11) is satisfied, provided $(\nu\lambda - \delta/2)$ is constant. Thus, by (2.7),

$$\left(\frac{r}{\alpha}\right)h'' + \left(\frac{r}{\alpha}\right)'h' - \nu\left[\left(\frac{r'}{\alpha}\right)h' + \left(\frac{r'}{\alpha}\right)'h\right] = \left(\frac{r}{\alpha}\right)Kf(\xi). \tag{2.16}$$

This equation is directly integrable and its solution is

$$h = Kr^{\nu}\int r^{-(1+\nu)}\alpha\left[\int\left(\frac{r}{\alpha}\right)f\,d\xi\right]d\xi + c_1r^{\nu} + c_2r^{\nu}\int r^{-(1+\nu)}\alpha\,d\xi, \tag{2.17}$$

where c_1, c_2 are constants of integration. It may be noted that setting $K = 0$ corresponds to the vanishing value of $(\nu\lambda - \delta/2)$ or, by (2.10), to $k = 1$.

(b) For shells of uniform thickness, δ vanishes identically and we have

$$\lambda = f^{-1}(\xi)\left[\frac{(r'/\alpha)'}{(r/\alpha)}\right], \qquad \delta = 0. \tag{2.18}$$

Clearly, λ is a function of ξ and its form is determined by the geometry of the middle surface. However, for numerous shell configurations λ and, by (2.10), k is either exactly or very nearly a constant.

It should be mentioned that whenever the radius of curvature of the generating curve r_1 is a constant (r_2 may be a function of ξ), then, with proper choice of ξ ($\xi = \phi$) and by (1.2), (2.4) and (2.18), λ, and thus k, are in fact constant. The cases of conical shell and toroidal shell, treated recently by Clark [2], are included in this class.

3. Remarks on the solution of equation (2.14). Particular solutions of Eq. (2.14) may be obtained approximately by the membrane theory of shells given by Hildebrand in [3] or sometimes by a more recent method developed by Clark and Reissner [4]. In this section, we shall discuss briefly the homogeneous solution of equation (2.14).

If Ψ^2 and Λ are suitably regular over a finite interval of the ξ-axis and furthermore, if Ψ^2 is bounded from zero everywhere within this interval, then the classical method of asymptotic integration leads at once to the solution

$$W = \Psi^{-1/2}\{Ae^{-i\eta} + Be^{i\eta}\} \tag{3.1}$$

which, by means of well-known relations for Bessel functions, may also be written as

$$W = \left[\Psi^{-1}\int\Psi\,d\xi\right]^{1/2}\{AJ_{-1/2}(\eta) + BJ_{1/2}(\eta)\}, \tag{3.2}$$

where A and B are constants and

$$\eta = (2i^3)^{1/2}\mu\int\Psi\,d\xi. \tag{3.3}$$

According to Langer [5], if Ψ^2 is not bounded from zero everywhere within the interval in question but vanishes to the degree n at some point ξ_0 within this interval, then (3.2)

may be generalized to

$$W = \left[\Psi^{-1} \int_{\xi_0}^{\xi} \Psi(\zeta)\, d\zeta \right]^{1/2} \{ AJ_{-1/n+2}(\eta) + BJ_{1/n+2}(\eta) \} \tag{3.4}$$

which is valid at ξ_0 .

In (3.4)

$$\eta = (2i^3)^{1/2} \mu \int_{\xi_0}^{\xi} \Psi(\zeta)\, d\zeta.$$

An appropriate form of this solution was recently employed for toroidal shells by Clark [2].

If, moreover, the coefficient function Λ of the differential equation has a pole of second order at ξ_0 , then by a more recent method of asymptotic integration developed by Langer [6], a representation of W in terms of Bessel functions is again possible and is valid at ξ_0 . It may be mentioned that application of this method yields a solution for ellipsoidal shells of revolution which is valid at the apex (where a pole of second order occurs); this will be given on another occasion.

4. Acknowledgement. The authors are indebted to a referee for helpful criticism.

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DISTRIBUTION OF THE EXTREME VALUES OF THE SUM OF n SINE WAVES PHASED AT RANDOM*

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1. Introduction. The statistical behavior of the sum of n sine waves phased at random has been studied in connection with a number of technical problems. These include radio wave fading and overloading in multichannel telephony.[1]

When the sine waves are of unit amplitude their sum may be written as

$$z = \sum_{m=1}^{n} \cos \varphi_m \tag{1.1}$$

where $\varphi_1, \varphi_2, \cdots, \varphi_n$ are independent random angles, each distributed uniformly over the range $-\pi$ to π. z cannot exceed n. When $n - 2 < z \leq n$ the probability density of z may be expressed as a power series in $(n - z)$, as is shown in Sec. 2.

There is a close relation between the distribution of z and the problem of the random walk in two dimensions, and the two are often treated together. Several equations connecting them are given in Sec. 3. In Sec. 4 the results of Sec. 2 are used to obtain the first few terms in a series for the distribution of the extreme values in the random walk problem.

When n is large the central portion of the distribution for z approaches a normal law. In Sec. 5 an attempt is made to obtain an approximation to the distribution over the entire range of z by interpolating between the normal law result for small z and the results of Sec. 2 which hold for extreme values of z. The work is carried out first for the random walk distribution and then translated to the z distribution. This procedure is used because the random walk distribution seems to be better suited to our method of interpolation than does the z distribution. Figure 1 is associated with the interpolation between the results given by Pearson[2] and Rayleigh[3] for the random walk and those of Sec. 4.

I wish to express my thanks for the many helpful suggestions concerning this paper which I have received from Mr. John Riordan and others.

2. Series for the probability density of z when z is near n. Let $q_n(z)$ denote the probability density of the random variable z defined by (1.1). Then, when $n - 1 <$

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[1]See, for instance, W. R. Bennett, Distribution of the sum of randomly phased components, Quarterly Appl. Math. **5**, 385-393 (Jan. 1948). References to earlier work will be found in this paper. Mention should also be made of papers by F. Horner, Phil. Mag. (7) **37**, 145-162 (1946), and R. D. Lord, Phil. Mag. (7) **39**, 66-71 (1948). The second paper gives our equation (3.5).

[2]Drapers' Co. Research Memoirs Biometric Series III. *Math. Contributions to the theory of evolution—XV. A mathematical theory of random migration*, Karl Pearson assisted by John Blakeman, London (1906).

[3]Rayleigh, Phil. Mag. **10**, 73 (1880) and Phil. Mag. **37**, 321-347 (1919).

$z < n + 1$,

$$q_{n+1}(z) = \int_{z-1}^{n} q_1(z-u)q_n(u)\,du$$
$$= \frac{1}{\pi}\int_{z-1}^{n} [1-(z-u)^2]^{-1/2} q_n(u)\,du. \tag{2.1}$$

The change of variables

$$x = n + 1 - z, \qquad u = z - 1 + vx$$

carries (2.1) into

$$q_{n+1}(n+1-x) = \frac{(x/2)^{1/2}}{\pi}\int_0^1 [v(1-vx/2)]^{-1/2} q_n[n-(1-v)x]\,dv \tag{2.2}$$

which holds when $0 < x < 2$. When we place the assumed expansion

$$q_n(n-x) = A_n x^{n/2-1}\left[1 + \sum_{k=1}^{\infty} a_{nk}(x/4)^k/n(n+2)\cdots(n+2k-2)\right] \tag{2.3}$$

in (2.2) and use

$$q_1(z) = q_1(1-x) = \pi^{-1}[2x-x^2]^{-1/2} \tag{2.4}$$

we obtain

$$A_n = (2\pi)^{-n/2}/\Gamma(n/2) \tag{2.5}$$

and a set of recurrence relations, the lth of which is

$$a_{n+1,l} = \sum_{k=0}^{l} a_{n,l-k}\alpha_k^2/k!, \qquad l = 0, 1, 2, \cdots, \tag{2.6}$$

where $a_{n0} = 1$, $\alpha_0 = 1$ and

$$\alpha_k = 1\cdot 3\cdot 5\cdots(2k-1).$$

The series in (2.3) converges for $|x| < 2$ as may be seen by substituting $q_n(n-x) = x^{-1+n/2}f_n(x)$, $f_1(x) = \pi^{-1}(2-x)^{-1/2}$ in (2.2). An integral is obtained which may be used to show in succession that $f_2(x), f_3(x), \cdots f_n(x)$ are analytic functions of x inside the circle $|x| = R$, $R < 2$, in the complex x-plane.

Equations (2.6) and $a_{1k} = \alpha_k^2/k!$ lead to

$$\sum_{k=0}^{\infty} a_{nk}s^k = \left[\sum_{j=0}^{\infty} \alpha_j^2 s^j/j!\right]^n. \tag{2.7}$$

This is merely a formal result because the series on the right does not converge.

It is interesting to note that (2.7) fits in with some heuristic manipulation of the integral

$$q_n(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-izt}[J_0(t)]^n\,dt. \tag{2.8}$$

Thus if we raise the asymptotic expression

$$J_0(t) \sim \sum_{m=0}^{\infty} \frac{\pi^{-1/2}\alpha_m^2}{m!4^m}\left[\frac{e^{it}}{(i2t)^{m+1/2}} + \frac{e^{-it}}{(-i2t)^{m+1/2}}\right] \tag{2.9}$$

(where $-\pi < \arg t < \pi$ and $\arg(-i) = -\pi/2$) to the nth power we obtain the sum of terms of the form $\exp i(n - 2l)t$ times a series in $1/t$ with $l = 0, 1, \cdots n$. Let the t-plane be cut along the negative real t axis and let the limits of integration in (2.8) be $-i \pm \infty$ instead of $\pm\infty$. When we substitute (2.9) in (2.8), assume $n - 2 < z < n$, and use

$$\int_{-i-\infty}^{-i+\infty} \frac{e^{ixt}}{(it)^{\nu}}\, dt = \begin{cases} \dfrac{2\pi x^{\nu-1}}{\Gamma(\nu)}, & x > 0 \\ 0, & x < 0 \end{cases}$$

only the terms multiplied by $\exp[i(n - z)t]$ (corresponding to $l = 0$, $x = n - z$) contribute to the value of the integral. Furthermore, they lead to the same series in x for $q_n(n - x)$ as does (2.7). If this procedure could be justified and generalized it might lead to an expression for (2.8) which would supplement the one obtained by the method used by W. R. Bennett.[1]

The coefficients a_{nk} may be expressed in terms of Bell's Y polynomials.* Thus, multiplying both sides of (2.7) by $t^n/n!$ and summing n from 0 to ∞ shows that a_{nk} is the coefficient of $t^n s^k/n!$ in

$$[\exp t]\left[\exp \sum_{j=1}^{\infty} \left(t\alpha_j^2 \frac{s^j}{j!}\right)\right] = e^t \sum_{k=0}^{\infty} \frac{s^k}{k!} Y_k(t\alpha_1^2, t\alpha_2^2, \cdots, t\alpha_k^2)$$

where the $Y_k(y_1, y_2 \cdots y_k)$ are Bell's polynomials:

$$Y_0 = 1, \qquad Y_1 = y_1, \qquad Y_2 = y_2 + y_1^2, \qquad Y_3 = y_3 + 3y_2y_1 + y_1^3, \cdots.$$

Consequently,

$$a_{nk} = \frac{n!}{k!} Y_k(t\alpha_1^2, t\alpha_2^2, \cdots, t\alpha_k^2), \tag{2.10}$$

where, after writing the right hand side as a polynomial in t, t^m is replaced by $1/(n - m)!$. We obtain in this way

$$q_n(n - x) = \frac{(x/2\pi)^{n/2-1}}{2\pi\Gamma(\frac{1}{2}n)}\left[1 + \frac{x}{4} + \frac{(n + 8)x^2}{32(n + 2)} + \frac{(n^2 + 24n + 200)x^3}{384(n + 2)(n + 4)} + \cdots\right], \tag{2.11}$$

where the series converges when $0 \leq x < 2$.

The probability $\Psi(E)$ that $|z| \geq E$ is given by

$$\Psi_n(E) = 2\int_E^n q_n(z)\, dz = 2\int_0^{n-E} q_n(n - x)\, dx \tag{2.12}$$

and when $n - 2 < E \leq n$ this leads to

$$\Psi_n(E) = \frac{2}{\Gamma[\frac{1}{2}(n + 2)]}\left[\frac{n - E}{2\pi}\right]^{n/2}\left[1 + \frac{n(n - E)}{4(n + 2)} + \frac{n(n + 8)(n - E)^2}{32(n + 2)(n + 4)} + \cdots\right]. \tag{2.13}$$

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*Actually what is used is the slightly more general version of these polynomials given by John Riordan, *Derivatives of composite functions*, Bull. Amer. Math. Soc. 52, 664-667 (1946).

3. Relation between z and the problem of the random walk. The random variable z, defined as the sum of n cosines by (1.1), may be regarded as the projection of the resultant r (of n unit random vectors) on the x-axis. Hence, we may write $z = r \cos \theta$ where θ is a random angle distributed uniformly over the interval $(0, 2\pi)$. The probability $p_n(r)\, dr$ that the length of the resultant lies between r and $r + dr$ is given by the random walk distribution when the n elementary linear walks are of unit length each. We shall use $\Phi_n(r)$ to denote the probability that the resultant equals or exceeds r. Then the connection between z and r leads to the following relations between the probability functions:

$$\Psi_n(E) = \frac{2}{\pi} \int_E^n p_n(r) \text{ arc cos } (E/r)\, dr, \tag{3.1}$$

$$\Psi_n(E) = \frac{2E}{\pi} \int_E^n r^{-1}(r^2 - E^2)^{-1/2}\Phi_n(r)\, dr, \tag{3.2}$$

$$\Phi_n(r) = -r \frac{d}{dr} \int_r^n (E^2 - r^2)^{-1/2}\Psi_n(E)\, dE, \tag{3.3}$$

$$\Phi_n(r) = r^2 \int_r^n E^{-1}(E^2 - r^2)^{-1/2}[E^{-1}\Psi_n(E) - \Psi_n'(E)]\, dE, \tag{3.4}$$

$$q_n(z) = \frac{1}{\pi} \int_z^n (r^2 - z^2)^{-1/2} p_n(r)\, dr, \tag{3.5}$$

$$p_n(r) = -2r \int_r^n (z^2 - r^2)^{-1/2}[dq_n(z)/dz]\, dz. \tag{3.6}$$

In these equations E, r and z are assumed to be less than n and $p_1(r)$ is to be interpreted as an impulse function. In (3.6) n must exceed two but this causes no difficulty since it is known that

$$p_2(r) = (2/\pi)(4 - r^2)^{-1/2}, \qquad -2 < r < 2. \tag{3.7}$$

In going from (3.1) to (3.2) we have integrated by parts. Setting $n^2 - r^2 = \xi$, $n^2 - E^2 = x$ in (3.2) converts it to a special case of Abel's integral equation* whose solution gives (3.3). The remaining equations are obtained by the same kind of analysis.

4. Random walk distributions when r is near n. When $n > 2$ and $n - 2 < r < n$, substitution of the expression (2.11) for $q_n(z)$ in the integral (3.6) for the probability density $p_n(r)$ of r gives

$$p_n(r) = \frac{n^{1/2}}{2\pi\Gamma[\frac{1}{2}(n-1)]} \left(\frac{n-r}{2\pi}\right)^{(n-3)/2} \left[1 + \frac{(n-1)(n-r)}{4n} + \frac{(n^2 + 4n - 9)(n-r)^2}{32n^2} + \cdots\right]. \tag{4.1}$$

When $n = 2$ the method fails, but it is not difficult to show from (3.7) that (4.1) also holds in this case. Expression (4.1) may also be obtained from the recurrence relation for $p_n(r)$ by a method similar to that used in Sec. 2 to obtain $q_n(n - x)$. Pearson[2] has

*See, for example, Whittaker and Watson, *Modern analysis*, 4th ed., Cambridge, 1927, p. 229.

given, essentially, the leading term in (4.1) and has given one or two more terms for $n = 3, 4, 5, 6$.

Integrating (4.1) termwise gives

$$\Phi_n(r) = \int_r^n p_n(\rho)\, d\rho = \frac{n^{1/2}}{\Gamma[\frac{1}{2}(n+1)]}\left[\frac{n-r}{2\pi}\right]^{(n-1)/2}\left[1 + \frac{(n-1)^2(n-r)}{4n(n+1)} + \frac{(n-1)(n^2+4n-9)}{32n^2(n+3)}(n-r)^2 + \cdots\right] \quad (4.2)$$

which holds for $n - 2 < r < n$.

5. Approximations for $\Phi_n(r)$ and $\Psi_n(E)$. Numerical values of the various probability densities and distributions have been given by Pearson[2], Slack[4], Bennett[1] and others for values of n up to 10 (and somewhat beyond in certain cases.) The values of $\Phi_n(r)$ and $\Psi_n(E)$ given by Bennett were computed from series which converge for all values of r and E between zero and n. In this section we shall consider a method of estimating values of $\Phi_n(r)$ and $\Psi_n(E)$ which involves only a small amount of calculation but which, of course, lacks the accuracy of the computations mentioned above.

When n is large, but E and r of moderate size, it is known that

$$\Phi_n(r) \approx \exp(-r^2/n), \quad (5.1)$$

$$\Psi_n(E) \approx 1 - \operatorname{erf}(E/n^{1/2}), \quad (5.2)$$

$$\operatorname{erf}(x) = 2\pi^{-1/2}\int_0^x \exp(-x^2)\, dx.$$

Our method of estimation is roughly equivalent to interpolating between values given by these formulas and those given by the formulas of Sec. 2 and 4. We shall first work with the random walk distribution.

Equation (5.1) suggests that we introduce a function y of r and n defined by

$$\Phi_n(r) = e^{-ny} \quad (5.3)$$

or

$$y = -\frac{1}{n}\log \Phi_n(r). \quad (5.4)$$

Comparison of (5.1) and (5.3) shows that, for small values of r/n, $y \approx (r/n)^2$ and hence in this case y depends "much more" on r/n than on n. The same is true when r/n is nearly unity since Eq. (4.2) for $\Phi_n(r)$ leads to (assuming n large so that the series may be approximated by $\exp[(n - r)/4]$)

$$y \approx -\frac{n-1}{2n}\log\left(1 - \frac{r}{n}\right) - \frac{1}{4}\left(1 - \frac{r}{n}\right) + \frac{1}{2}\log\frac{\pi}{e}, \quad (5.5)$$

and this again tends to become a function of r/n only as $n \to \infty$.

In order to test the dependence of y on r/n, Bennett's values of $\Phi_n(r)$ for $n = 6$ and 10 were used to compute y from (5.4). The results are plotted as the "exact" values of y

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[4] Margaret Slack, *The probability distributions of sinusoidal oscillations combined in random phase*, J.I.E.E., Pt. III, **93**, 76-86 (1946).

(indicated by the small triangles and circles) shown in Fig. 1. It is seen that the two sets of values tend to follow a common curve.

The dashed curves in Fig. 1 were computed from the approximation (5.5) for $n = 6, 10$ and ∞. Although (5.5) is valid only for $r/n \approx 1$, it yields values of y which are fairly

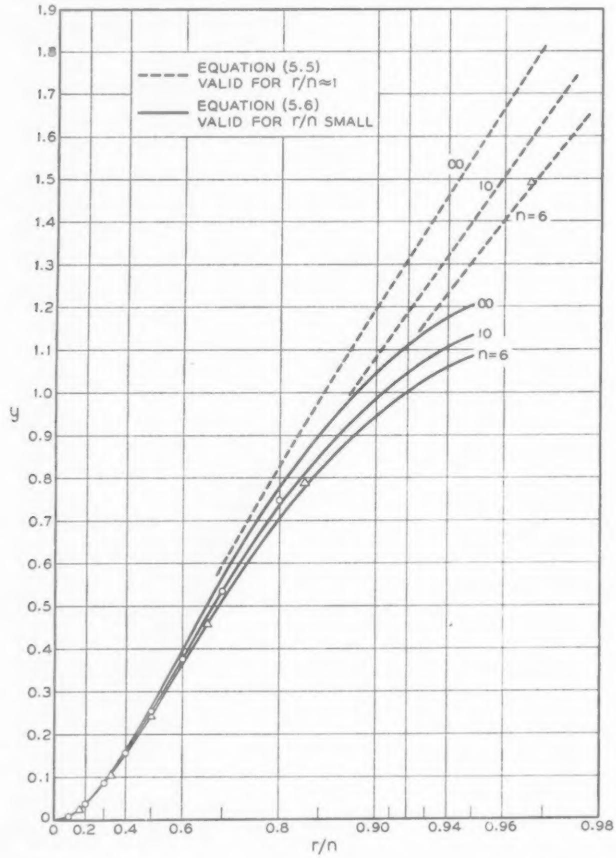

Fig. 1. The triangles ($n = 6$) and the circles ($n = 10$) show exact values of $y = (-1/n) \log_e \Phi_n(r)$. $\Phi_n(r)$ is the probability that the resultant of n random two-dimensional unit vectors is longer than r. The curves show the approximations (5.5) and (5.6) for $n = 6, 10$, and ∞.

close to the exact values even for values of r/n as small as 0.4. The solid curves were computed from

$$y \approx \left(\frac{r}{n}\right)^2 + \frac{1}{4}\left(\frac{r}{n}\right)^4 + \frac{5}{36}\left(\frac{r}{n}\right)^6 - \frac{1}{n}\left[\frac{1}{2}\left(\frac{r}{n}\right)^2 + \frac{1}{3}\left(\frac{r}{n}\right)^4\right] + \frac{1}{12n^2}\left(\frac{r}{n}\right)^2 \tag{5.6}$$

which holds only for small values of r/n and large values of n. Expression (5.6) is obtained

from

$$\Phi_n(r) \approx e^{-x}\left[1 - \frac{f_1}{2n} - \frac{2f_2}{3n^2} + \frac{(6n-11)f_3}{8n^3} + \cdots\right], \tag{5.7}$$

where $x = r^2/n$ and

$$f_m = -x\,{}_1F_1(-m;2;x) = -x + \frac{m}{1!2!}x^2 - \frac{m(m-1)}{2!3!}x^3 + \cdots,$$

${}_1F_1(\ )$ being a confluent hypergeometric function. Expression (5.7) may be obtained by integrating a result given by Pearson[2] and, later, by Rayleigh[3]. Pearson's formula suggests that the next two terms in (5.7) are $+ (50n - 57)f_4/15n^4 - (270n^2 - 2125n + 1892)f_5/144n^5$.

Thus, a rough idea of how $\Phi_n(r)$ behaves for all values of r/n and a particular value of n may be obtained by (i) computing approximations to y from (5.5) and (5.6),* (ii) plotting them on semi-log paper as shown in Fig. 1, (iii) joining the two portions by a smooth curve, and (iv) using these approximate values of y to compute $\Phi_n(r)$ from (5.3).

When we turn to $\Psi_n(E)$, the procedure used in dealing with $\Phi_n(r)$ suggests that a new function y' of E and n be defined by

$$\Psi_n(E) = 1 - \operatorname{erf}\,[(ny')^{1/2}]. \tag{5.8}$$

However, it is found that the expression for y' when $E/n \approx 1$ does not have the simplicity of its analogue (5.5). Instead of following this line of thought further, we note that the analogy between y and y' suggests that y' should not differ greatly from the function obtained by replacing r by E in y. That the difference is small may be verified by comparing Bennett's exact values of $\Psi_{10}(E)$ with approximate ones obtained from (5.8). In using (5.8), the values of y' are taken to be those of y as computed from (5.4) and Bennett's exact values of $\Phi_{10}(r)$. The exact and approximate values are shown in the following table for three values of E.

E	exact $\Psi_{10}(E)$	approx. $\Psi_{10}(E)$
1	.6604	.6624
5	.02337	.02440
8	.00009	.00011

Thus, once we have obtained approximate values of y from curves of the type shown in Fig. 1, approximate values of $\Phi_n(r)$ and $\Psi_n(E)$ may be obtained readily from (5.3) and (5.8) (with y in place of y').

*Instead of (5.6) one may use (5.7) and (5.4). This gives more accurate values of y at the cost of more computation.

THE MECHANICS OF THE RIJKE TUBE*

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1. Introduction. An interesting phenomenon, the analysis of which has received only scattered attention, is the thermal-acoustic oscillation discovered by Rijke in 1859. It was noted, under certain circumstances, that superimposed on the anticipated steady

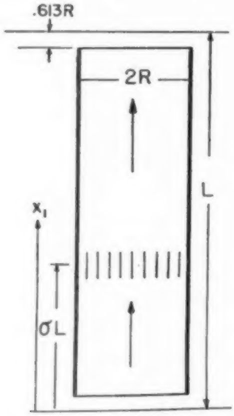

FIG. 1.1. Open ended cylindrical tube containing heated ribbon at $x_1 = \sigma L$.

convective flow in the apparatus indicated in Fig. (1.1) was an acoustic oscillation. The frequency of the phenomenon is essentially that of the first free oscillation mode of the pipe (i.e. the wave length is $2L'$),[1] but the occurrence (or lack thereof) and the intensity depend on a number of parameters.

The analysis of the effect of these parameters which is presented here proceeds from the fundamental conservation laws with an investigation of the response of the heater to a fluctuating velocity, an analysis of the non-isentropic wave propagation in the pipe with its attendant viscous losses and end radiation, and, from the interaction of these, the deduction of the complex eigenfrequency of the tube. In a more qualitative manner, the consistent appearance of harmonics is explained and the role these waves play in determining the sustained intensity of the sound is discussed.

The results are in excellent agreement with the observed physical facts, and it is believed that they, as well as the details of the analysis, may be useful in the future investigation of various combustion oscillation phenomena.

2. The wave propagation in the tube. It is convenient to divide the analysis of this composite problem into various pieces. The description of the waves in the tube for

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[1]L' is the length of the pipe corrected for end effects and non-uniform temperatures.

$x < \sigma$ and those for $x > \sigma$ are treated individually.[2] The boundary conditions at σ "joining" these waves are treated as though a disk shaped heat source, whose response to local velocity fluctuations is known, were located at σ. However, in the detailed investigation of the heater response, a more realistic description of the flow past the heater will be adopted.

In analyzing the oscillating flow downstream of the heater we must anticipate that it is non-isentropic and, in particular, that important viscous losses occur at the wall and that decaying temperature fluctuations are convected from the heater toward the exit. We assume that the steady flow upon which the oscillation is to be superimposed is known and introduce a small perturbation (in velocity, pressure, density, etc.) to represent the oscillatory phenomenon. The conservation laws (mass, momentum, and energy) are

$$(\rho' u_i')_{,i} + \rho'_{,t} = 0, \tag{2.1}$$

$$\rho'(u'_{i,t} + u_j' u'_{i,j}) + p'_{,i} = \mu(u'_{i,jj} + \tfrac{1}{3}u'_{j,ij}) + F_i\,, \tag{2.2}$$

$$\rho' c_v(T'_{,t} + u_i' T'_{,i}) - (p'/\rho')(\rho'_t + u_i'\rho'_{,i}) - k\Delta T' = 0, \tag{2.3}$$

and we adopt the state equation

$$p' = \rho' R' T'. \tag{2.4}$$

Here, p', ρ', T', u_i' are the usual thermodynamic variables and the velocity, and F_i is the gravitational body force. It is convenient and appropriate in this analysis to treat c_p, c_v, μ, k, as though they were independent of the thermodynamic state. It is also convenient to introduce the notation[3]

$$\begin{aligned}
u_i' &= v_i(x, r) + a[\operatorname{grad}\varphi(x, r, \tau) + \operatorname{curl}\mathbf{k}\psi(x, r, \tau)],\\
p' &= p_0(x, r)[1 + p(x, r, \tau)],\\
\rho' &= \rho_0(x, r)[1 + \eta(x, r, \tau)],\\
T' &= T_0(x, r)[1 + \theta(x, r, \tau)],\\
x &= x_1/L, \qquad r^2 = (x_2^2 + x_3^2)/L^2, \qquad a^2 = \gamma p_0/\rho_0\,, \qquad \tau = at/L,
\end{aligned} \tag{2.5}$$

where $\mathbf{k}$ is the unit vector perpendicular to both the x and r directions. The tensor notation differentiations of Eqs. (2.1) to (2.3) are taken with regard to the physical coordinates (the x_i), but the vector notation differentiations of Eq. (2.5) pertain to the variables x, r. In the definitions of x, r, τ, the constant length parameter L may be thought of as the length of the tube. It will be defined more precisely later.

Equations (2.5) are now substituted into Eqs. (2.1) to (2.4), the terms containing no perturbation contributions are removed (since the steady terms are themselves solutions of the conservation equations), and the remaining equations for the perturbation quantities are linearized. Furthermore, the following simplification is adopted:

[2]See Fig. (1.1).

[3]This notation is appropriate for either the region upstream of σ or that downstream of σ. We shall distinguish these in Sec. 4 by subscripts 2 and 1 respectively.

p_0, ρ_0, T_0, v_0 are replaced by appropriate constant "average" values (in particular $v_1 = Ma$) and v_2, v_3 are taken to vanish. The resulting perturbation equations are:

$$\eta_{,\tau} + \Delta\varphi = 0, \tag{2.6}$$

$$\varphi_{,\tau} + \gamma^{-1}p = \tfrac{4}{3}\epsilon\,\Delta\varphi, \tag{2.7}$$

$$\theta_{,\tau} - (\gamma - 1)\eta_{,\tau} = \tfrac{4}{3}\gamma\epsilon\,\Delta\theta, \tag{2.8}$$

$$\psi_{,\tau} = \epsilon\,\Delta\psi, \tag{2.9}$$

$$p = \eta + \theta. \tag{2.10}$$

Here we have taken the Prandtl number to be 3/4 for algebraic simplicity; ϵ is defined by $\mu/\rho aL$ and $\gamma = c_p/c_v$. The operator denoted by $(\ )_{,\tau}$ implies $\partial(\ )/\partial\tau + M\partial(\ )/\partial x$. That is, it is the convective time derivative.[4]

These equations must be solved subject to the boundary conditions that, at the wall, the temperature and velocity fluctuations vanish, and at the plane $x = \sigma$, the velocity, temperature, etc. are consistent with the heater behavior. At the exit, and at the inlet, a reflection coefficient must be adopted but we shall discuss that later. Equations (2.6) to (2.10) may be integrated in detail subject to such boundary conditions. The simplest presentation of the solution is obtained by eliminating η, p, and θ, and by anticipating that the form of the solutions is exponential in x and τ. For example, $\psi(x, r, \tau) = \psi(r) \exp [i(\alpha\tau - kx)]$. Actually we must anticipate that there will be two modes of propagation; one which is essentially an acoustic wave and one which drifts with the stream. The former should consist principally of a contribution from φ where $k^2 \simeq \alpha^2$ (this implies that it propagates with a phase velocity which is essentially the acoustic speed), whereas the latter requires a less elementary description. The acoustic wave will consist of two parts; one which propagates in the downstream direction and a reflection from the tube end which propagates back upstream.

Using the same letters to denote the functions of interest with the exponential dependence factored off, we obtain the following ordinary differential equations for φ and ψ after eliminating η, p, and θ.

$$L_1L_2(\varphi) = 0, \qquad L_3(\psi) = 0, \tag{2.11}$$

where $L_1 = \Delta + \beta^2/(1 + 4i\beta\gamma\epsilon/3)$, $L_2 = \Delta - 3i\beta/4\epsilon$, $L_3 = \Delta - i\beta/\epsilon$, $\Delta = r^{-2}\partial^2/\partial r^2 + r^{-1}\partial/\partial r - k^2$, and $\beta = \alpha - kM$.

For the upstream section analysis, it follows from our earlier remarks that $\beta = \nu\alpha - kM$. The propagation modes are found by applying the homogeneous boundary conditions at the cylindrical wall and finding the eigenvalues k. If we anticipate that $\varphi = \varphi_1 + \varphi_2$ with $L_1(\varphi_1) = L_2(\varphi_2) = 0$ for the acoustic mode, and further anticipate that in this mode φ_2 and ψ will each display a boundary layer character[5] whereas φ_1 will essentially be a plane acoustic wave, the motion is described to terms of order ϵ by

$$\varphi(r) = AJ_0[(\beta^2 - k^2)^{1/2}r] + B \exp [-(r_1 - r)(3i\beta/4\epsilon)^{1/2}] \tag{2.12}$$

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[4]When we consider the waves in the upstream region, it will again be convenient to define $\tau = at/L$ giving a downstream value. The definition of $(\ )_{,\tau}$ then will become $\nu\partial/\partial\tau + M\partial/\partial x$ where $\nu = a_1/a_2$ and M is evaluated for the upstream quantities.

[5]The term boundary layer is used here in the more general sense as in [1].

and

$$\psi(r) = C \exp\left[-(r_1 - r)(i\beta/\epsilon)^{1/2}\right], \tag{2.13}$$

where $r_1 = R/L$. Since $\theta(r) = p - \eta = [-i\gamma\beta + (i\beta)^{-1}\Delta]\varphi(r)$, the characteristic equation for k becomes [using $\varphi_r(r_1) + \psi_x(r_1) = \psi_r(r_1) - \varphi_x(r_1) = \theta(r_1) = 0$],

$$\begin{vmatrix} -ik & -ik & (i\beta/\epsilon)^{1/2} \\ r_1(k^2 - \beta^2)/2 & (3i\beta/4\epsilon)^{1/2} & ik \\ i(\gamma - 1)\beta & -3/4\epsilon & 0 \end{vmatrix} = 0, \tag{2.14}$$

or (again to order ϵ)

$$r_1(k^2 - \beta^2)(i\beta/\epsilon)^{1/2} = 2[k^2 + (\gamma - 1)(4/3)^{1/2}\beta^2]$$

and

$$k = \pm\alpha\left[1 + \frac{(\gamma - 1)(4/3)^{1/2}}{r_1}(\epsilon/i\beta)^{1/2}\right]. \tag{2.15}$$

The imaginary part of this number defines the decay rate of the wave which is associated with the dissipation and heat conduction near the wall. Its order of magnitude for many experiments (in particular, ours), renders it an important source of energy loss.

The propagation of the other "drifting" wave cannot be expressed in as elementary a form. However, we may obtain a reasonably simple representation of this wave if we anticipate the nature of the result. It is clear that the wave would be a plane wave drifting with the stream if it were not for the cylindrical wall boundary conditions. The effect of the walls, however, is an encroachment on this wave of such a nature that one cannot expect a simple product type representation corresponding to the acoustic wave. If one attempts the product type representation, an infinite series of such products is required. For an efficient representation, one should formulate an initial value problem in x such that the temperature has a uniform value (no dependence on r) at $x = \sigma$ but such that $\mathbf{v}$ and θ vanish at the wall. The solution to such a problem to the order of accuracy we require is given by $\psi = 0$ and

$$\varphi^{(3)}(x, r) = \frac{\partial}{\partial x}\int_0^{r_1 - r} e^{qx}K_0[\zeta(x^2 + S^2)^{1/2}]\, dS, \tag{2.16}$$

where $q = 3M/8\epsilon$ and $\zeta = (q^2 + 3i\alpha/4\epsilon)^{1/2}$. A more accurate but no more useful (corrections of order $\epsilon^{1/2}$) solution would again involve boundary layer solutions in ψ and φ in addition to a slightly modified $\varphi^{(3)}$.

Equation (2.16) is valid only for small enough x so that $\varphi^{(3)}(x, 0)$ takes essentially the value $\varphi^{(3)}(x, -\infty)$. However, for those x violating this condition, the wave has decayed so far as to be of no further interest. The critical observation concerning these eigenmodes of the tube is that the mode associated with $\varphi^{(3)}$ provides velocity and pressure contributions which are very small compared with its temperature and density contributions. Conversely, the k_1, k_2, modes give contributions of the same order in each of these four (dimensionless) state variables. This implies that the velocity and pressure fluctuations are almost entirely contributed by the acoustic modes while the final mode merely picks up any discrepancy in the temperature condition at $x = \sigma+$. We shall see this more clearly in Sec. 4.

The wave in the upstream portion of the tube contains only the acoustic modes. The other possible mode is again a downstream drifting wave which has zero "input" in the upstream section of the tube and is therefore not present. The formulas and k values for this section are identical with those of the downstream section except that the acoustic velocity, Mach number, M, etc. must be the values for the upstream state instead of the downstream state. In particular α is replaced everywhere by $\nu\alpha$ except in the exponential exp $(\alpha\tau)$.

Before closing this section, we must discuss the relation between the outgoing (away from the heater) waves and the reflected returning waves. The reflection coefficient for an acoustic wave in an unflanged tube has been investigated by Levine and Schwinger [2] for the case $M = 0$. We have shown, and shall report in detail elsewhere, that the following is true. If the plane axially directed wave incident on the exit end of a tube (from inside) has acoustic pressure $p_i = P \exp(i\omega t)$ at some point x, the reflected wave has pressure $p_r = PN' \exp(i\omega t)$ where $N'(\omega R/a, M) \equiv N[\omega R/a(1 - M^2)^{1/2}]$. The formula for the inlet end is $p_r/p_i = (1 - M/1 + M)N[\omega R/a(1 - M^2)^{1/2}]$. Here N is the reflection coefficient computed in [2] for $M = 0$. However, for $S \ll 1$, $N(S) = 1 - S^2/2 + \cdots$, and arg N is such that the wave is apparently reflected with reflection coefficient $|N|$ from a point $.613R$ beyond the end of the tube.[6] Since M is of order 10^{-3} for our problem, the moving gas correction is negligible and we use the above results with $S = wr/a$ (but with differing values of "a" in the two sections of the tube). We now define L more precisely. In fact $x_1 = 0$ is the point $.613R$ below the upstream end of the tube and L is the point $.613R$ beyond the other end. We shall return to, and use these results when we have determined the conditions at $x = \sigma$ which are imposed by the heater. To facilitate this, we record, respectively, the downstream (of σ), and upstream, acoustic wave forms for use in Sec. 4, replacing J_0 of argument whose order is 10^{-3} by unity.

$$\varphi^{(1)} = A_1 \exp[i\alpha\tau - ik_1(x-1)] - |N_1| \exp[i\alpha\tau + ik_1(x-1)], \tag{2.17}$$

$$\varphi^{(2)} = A_2 \exp(i\alpha\tau + ik_2 x) - |N_2| \exp(i\alpha\tau - ik_2 x), \tag{2.18}$$

with

$$k_1 = \alpha\left[1 + \frac{1 + (\gamma - 1)(4/3)^{1/2}}{r_1}\right](\epsilon/i\alpha)^{1/2} = k_1' - ik_1'',$$

$$k_2 = \nu\alpha\left[1 + \frac{1 + (\gamma - 1)(4/3)^{1/2}}{r_1}\right](\epsilon/i\nu\alpha)^{1/2} = k_2' - ik_2'',$$

$$N_1 = 1 - (\alpha r_1)^2/2 + \cdots,$$

$$N_2 = 1 - (\nu\alpha r_1)^2/2 + \cdots,$$

$$\nu = a_1/a_2 .$$

The k_1', k_1'', separation is such that k_1' is real for real α.

To the order of accuracy to which it is sensible to work, the downstream fluctuating velocity and pressure are given by

$$u^{(1)} = a_1\varphi_x^{(1)}, \qquad p^{(1)} = -i\alpha\gamma\varphi^{(1)} \tag{2.19}$$

[6]In our problem S is of order 0.1.

and, correspondingly,

$$u^{(2)} = a_2\varphi_x^{(2)}, \qquad p^{(2)} = i\nu\alpha\gamma\varphi^{(2)}. \tag{2.20}$$

As we shall see, the explicit formulas for the contributions of $\varphi^{(3)}$ will not be needed.

3. The heater "response". In this section[7] we shall estimate the fluctuating heat release from a heated ribbon to a stream of fluid whose velocity is fluctuating in a known manner. We say "estimate" because certain simplifying approximations will be adopted to render the problem tractable. However, the errors introduced should be of the order of a few per cent. In particular, we shall consider the problem whose simplified geometry

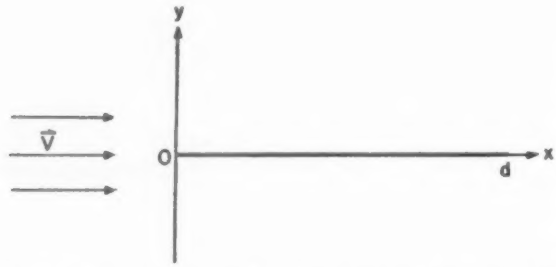

FIG. 3.1. The geometry of the simplified heater response problem. The ribbon thickness is oriented in the y direction; $0d$ is its breadth. **V** represents the fluctuating flow.

is described in Fig. 3.1. For a known flow of an incompressible fluid with constant thermal conductivity and specific heat, the energy conservation equation has the form

$$\rho c_p(T'_t + \mathbf{v}\cdot\operatorname{grad} T') - k\,\Delta T' = 0. \tag{3.1}$$

Since the velocity distribution near the plate is rather complicated, this equation would be difficult to treat in its present form. However, the analysis of a similar problem [3], that of determining the viscous flow past an obstacle at low Reynolds number[8], has indicated that an excellent approximation to the result is obtained when one replaces $\mathbf{v}\cdot\operatorname{grad} T'$ by cUT'_x in Eq. (3.1). Here c is a number, $0 < c < 1$, (.43 is the optimum value in the viscous flow problem), U the free stream velocity, x the coordinate oriented in the flow direction and $T'(x, y, t)$ the fluid temperature. We introduce the notation $cU = u_0 + we^{i\tau}$, $T' = \Theta(x, y) + \theta(x, y)e^{i\tau}$, and T_0 (the plate temperature) $= T + \delta e^{i\tau}$. We shall take T and δ to be constant rather than functions of distance along the plate. This is not correct in detail but two items justify this choice. A precise analysis of the surface temperature of the wire is very messy and the correction to the heat release

[7]Unfortunately, the number of symbols needed in this paper is so large that the quantities a, β, and other notations of this section are not related to those labeled with the same letters in other sections. This should lead to no confusion since the equations are never directly combined.

[8]The nature of this approximation is discussed in [3], [4], and its verification noted in [4]. The present problem is very closely related to these and it is to be anticipated that the results obtained for the macroscopic quantities should be highly accurate. The reader should note, in particular, that the predicted steady heat release is in agreement with the known information concerning the heat release from plates to fluids.

rate associated with this approximation is not an important contribution to the total release[9].

The differential equation may now be "separated" into two equations governing, respectively, the steady and fluctuating stream temperatures provided we assume that each fluctuating quantity is small compared to the corresponding steady quantity (e.g. $\theta/\Theta \ll 1$). These equations take the form

$$\Delta\Theta - a\Theta_x = 0, \tag{3.2}$$

$$\Delta\theta - a\theta_x - i\beta\theta = (aw/u_0)\Theta_x\,, \tag{3.3}$$

where the second order term in $w\theta_x$ has been omitted. In these equations $a = \rho c_p u_0/k$ and $\beta = \rho c_p \omega/k$.

The boundary conditions are

$$\Theta \to 0 \quad \text{as} \quad \mathscr{I}m\,(x + iy)^{1/2} \to \infty, \qquad \Theta(x, 0) = T \quad \text{on the plate,}$$

and

$$\theta \to 0 \quad \text{as} \quad \mathscr{I}m\,(x + iy)^{1/2} \to \infty, \qquad \theta(x, 0) = \delta \quad \text{on the plate.}$$

We consider first the problem of determining $\Theta(x, y)$ when d(Fig. 3.1) is indefinitely large. This problem is readily solved by invoking Fourier Transforms and the Wiener-Hopf technique. Omitting the details of the analysis, the reader can readily verify that $\boldsymbol{\Theta}(\xi, y)$ is the Fourier Transform [10] with regard to x of the required $\Theta(x, y)$.

$$\boldsymbol{\Theta}/T = \frac{a}{i\xi(a - i\xi)} \exp\{-\mid y \mid [i\xi(a - i\xi)]^{1/2}\}. \tag{3.4}$$

The inversion integral is indented to pass beneath the origin (in the ξ plane). In particular, the transform of Θ_x/T [(which will be needed to solve Eq. (3.3)] is

$$\boldsymbol{\Theta}_x/T = [a/(a - i\xi)] \exp\{-\mid y \mid [i\xi(a - i\xi)]^{1/2}\}. \tag{3.5}$$

Other useful results are:

$$\Theta_y(x, 0)/T = -(a/\pi x)^{1/2}, \qquad x > 0 \tag{3.6}$$

$$\Theta(x, 0)/T = \operatorname{erfc}[(-ax)^{1/2}], \qquad x < 0 \tag{3.7}$$

and

$$Q_0 = 2kl \int_0^d \Theta_y(x, 0)\, dx = 4klT(ad/\pi)^{1/2}. \tag{3.8}$$

This last quantity[11] will be associated with the steady heat output once we identify d with the true wire "breadth" and show that the contributions to the heat release rate of the finite plate is essentially that of the semi-infinite plate in the region $0 < x < d$. To see this we note that the heat release rate of the problem just completed, which in the physical problem should be zero for $x > d$, is given by Eq. (3.6). We can now solve the problem where we postulate that Θ_y'/T on $x > d$, $y = 0$ is $\mid y \mid (a/\pi x)^{1/2}/y$ and Θ'

[9] This has been analyzed crudely but will not be reported here in detail.

[10] $\boldsymbol{\Theta}$ is defined as $\int_{-\infty}^{\infty} \exp(-i\xi x)\, \Theta(x, y)\, dx$.

[11] l is the length of the wire perpendicular to the plane of Fig. 3.1.

vanishes for $x < d$. This is not quite the boundary value problem the superposition of which on the preceding one gives the correct answer, but if it turns out in this problem that Θ_y' is extremely small on $x < 0$, then the error in using the superposition of these two problems will be negligible. Actually, we anticipate that the change in heat release rate on $0 < x < d$, as well as that on $x < 0$, will be negligibly small and so we replace the boundary condition on Θ_y' in our superposition problem by the requirement that Θ_y'/T be $|y| (a\pi/d)^{1/2}/y$. If the heat release on $x < d$ associated with the solution to this problem is small, that associated with the properly formulated problem will be smaller. The solution of this problem may be treated like the previous one and the essential part of the result is:

$$(\pi d/a)^{1/2}\Theta_y'(x, 0)/T = \operatorname{erfc} [a(d - x)]^{1/2} - [\pi a(d - x)]^{-1/2} \exp [-a(d - x)], \qquad x < d. \tag{3.9}$$

In our problem, a is of the order of 50 cm^{-1} and this analysis for the finite plate is justified provided d is of the order .1 cm or so. The heat release from the plate associated with Θ' is implied by

$$Q_0' = 2kl \int_0^d \Theta_y'(x, 0)\, dx = lkT/(\pi a d)^{1/2}.$$

That is, for d of order .1 cm or more, Q_0' is of order $.05Q_0$ and can be safely neglected. Thus Eq. (3.8) defines the steady component of the heat release rate.

We now turn to Eq. (3.3) and the boundary conditions on $\theta(x, y)$, again for d indefinitely large. Another application of the same technique gives

$$\begin{aligned}\theta(\xi, y)/T = {} & [a/(a - i\xi)]^{1/2} \exp \{-|y| [i\xi(a - i\xi)]^{1/2}\} \\ & - [a/(a_1 - i\xi)]^{1/2} \exp \{-|y| [(i\xi + a_2)(a_1 - i\xi)]^{1/2}\} iaw/\beta u_0 \\ & + [\delta a_1/i\xi(a_1 - i\xi)T] \exp \{-|y| [(i\xi + a_2)(a_1 - i\xi)]^{1/2},\end{aligned} \tag{3.10}$$

where a_1, $-a_2$, are the roots of $z^2 - az - i\beta = 0 (a_1 \to a$, as $\beta \to 0)$. In particular,

$$\begin{aligned}\theta_y(x, 0)/T = {} & [i(a^3/\pi x^3)^{1/2}\, w/2\, u_0\beta][1 - \exp(-a_2 x)] \\ & + [(a_1)^{1/2}\, \delta/T][(\pi x)^{-1/2} \exp(-a_2 x) + a_2^{1/2} \operatorname{erf} (a_2 x)^{1/2}],\end{aligned}$$

and

$$\begin{aligned}2kl \int_0^d \theta_y(x, 0)\, dx = {} & \frac{2ia^{3/2}klTw}{\beta u_0} [(a_2)^{1/2} \operatorname{erf} (a_2 d)^{1/2} + (\pi d)^{1/2}(e^{-a_2 d} - 1)] \\ & + \delta kl(a_1/a_2)^{1/2}[(a_2 d + \mathrm{d}) \operatorname{erf} (a_2 d)^{1/2} + (a_2 d/\pi)^{1/2} e^{-a_2 d}].\end{aligned} \tag{3.11}$$

Again, we can estimate the correction required by the fact that the semi-infinite plate treatment is only approximate but again this correction is smaller than is consistent with the accuracy of our basic model.

In order to deduce the surface temperature fluctuation δ we must analyze the flow of heat within the wire. A very crude analysis of this item will demonstrate that the δ contribution is negligible. This rough analysis is conducted as follows. We consider an infinitely long slab of thickness $2b$ with surface temperature $\delta e^{i\tau}$. The differential equation to be solved (conservation of energy) is

$$K_w \theta_{yy}^{(w)} - \rho_w C_w \theta_t^{(w)} = 0$$

and the boundary conditions are

$$\theta^{(w)}(\pm b, \tau) = \delta e^{i\tau}.$$

Here, (w) merely denotes wire.

The solution can readily be obtained and, in particular,

$$\theta_y^{(w)}(b, \tau) = \Omega\delta \tanh (\Omega b) e^{i\tau}, \tag{3.12}$$

where

$$\Omega = (i\omega\rho_w C_v/K_w)^{1/2}.$$

The heat release rate for a wire of length l and breadth d is given by $Q' = 2K_w l d\theta_y^{(w)}(\mathrm{b})$, omitting the $e^{i\tau}$. This must be equated to the heat release to the fluid as given by Eq. (3.12) in order to determine δ. For ribbons of thickness of order .03 cm and breadth .1 cm (within rather large factors), δ is so small that "the δ term" of Eq. (3.11) is of order 10^{-2} or less times "the w term."

Thus the fluctuating heat release from the ribbon is essentially given by

$$q = -2kl \int_0^d \theta_y(x, 0)\, dx = \frac{-2ia^{3/2}klTw}{\beta u_0} [(a_2)^{1/2} \operatorname{erf} (a_2 d)^{1/2} + (\pi d)^{1/2}(e^{-a_2 d} - 1)]. \tag{3.13}$$

It should now be noted that (a_2/a) becomes imaginary and small when $\beta/a^2 \to 0$ but has angle $\pi/4$ and becomes large when $a^2/\beta \to 0$. This implies that for large β/a^2 the bracketed quantity in Eq. (3.11) is nearly a pure imaginary and the phase lag between the heat release rate fluctuation and the velocity fluctuation is small.[12] Conversely, when β/a^2 is small, the phase lag approaches $3\pi/8$. For the conventional Rijke tube experiment, the phase lag is of order $3\pi/8$.

Before we abandon the discussion of the heater behavior, we should discuss qualitatively the large amplitude behavior. To do this, consider first the extreme case where the free stream velocity is zero. In this case the response of the heater to a positive velocity fluctuation must be identical with its response to a negative velocity fluctuation. This is a direct consequence of the symmetry of the situation. Thus while the velocity fluctuation traverses one period, the heater is aware only of the absolute value of the velocity fluctuation and thus executes two cycles of heat release fluctuation. In other words, an input velocity fluctuation of frequency ω excites a heat release fluctuation of frequency 2ω. If the phenomenon is not linear, the higher harmonics will also be present. If one now estimates the expected behavior where w/u_0 is not small and where u_0 is not zero, he must conclude that the input signal at frequency ω gives rise to a heat release fluctuation with both ω, 2ω, and probably higher order components. The ratio of the 2ω content to the ω content of the output should be an increasing function of w/u_0. This information will be of use when we try to explain the harmonic content and sound level of the Rijke tube output.

4. The matching conditions at the heater. The problem of combining the results of the foregoing sections is now a relatively straightforward matter. We merely write down the requirement that mass, momentum and energy be conserved across the heater. If we denote the quantities of interest at the heater inlet by the usual symbols with

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[12]This corresponds to the situation conventionally encountered in hot-wire instrumentation problems.

subscript 2 and those at the exit with subscript 1, these laws take the form

$$\rho_1 u_1 - \rho_2 u_2 = -\int_0^X \rho_t \, dx, \tag{4.1}$$

$$\rho_1 u_1^2 + p_1 = \rho_2 u_2^2 + p_2 - D/\pi R^2 - \int_0^X (\rho u)_t \, dx, \tag{4.2}$$

$$\rho_1 u_1 c_p T_1 - \rho_2 u_2 c_p T_2 = Q/\pi R^2 - \int_0^X (\rho c_p T)_t \, dx, \tag{4.3}$$

where Q is the heat input rate to the gas from the ribbon and D is the drag of the ribbon. It is tacitly assumed in these equations that 0 and X are in those regions where conduction is not important, and it can readily be verified that the omitted kinetic energy terms are of higher order than we have consistently retained. In limits of the integrals in these equations, 0 and X are appropriately chosen points just upstream and downstream of the heater. The integral term of Eq. (4.1) can be expected to be of the same order of magnitude as the other terms in that equation and a careful estimate of its value is required. As we shall see, the value of X is most readily chosen after we obtain the information which allows us to circumvent the necessity of finding ρ_t in $0 < x < X$. Before proceeding with this, however, we note that the integral terms in the other equations are of higher order in M than the dominating terms and a few simple estimates lead to the results

$$p_1' = p_2', \tag{4.4}$$

$$u_1' = u_2'(1 + \lambda), \tag{4.5}$$

with $\lambda = (\gamma - 1)q/\gamma\pi R^2 p_0$, when q is defined by Eq. (3.13). The primes indicate the time-dependent components of the state variables and velocities. To obtain a more useful form for Eq. (4.1), we write the conservation equations for a heat conducting gas in a passage assuming a given heat input rate[13] $Q/\pi R^2$ and a one-dimensional flow. For brevity, we accept the implication of Eq. (4.4) that $p = p(t)$ only, in the x range of interest (i.e. $0 < x < X$), rather than obtain this result again from the momentum equation in differential form. The other conservation equations (mass and energy) are [with $p = p(t)$]

$$(\rho u)_x + \rho_t = 0, \tag{4.6}$$

$$\rho u h_x + \rho h_t - (k/c_p)h_{xx} = Q/\pi R^2, \tag{4.7}$$

where h is the enthalpy.

Writing $\rho = \rho_0(x) + \rho'(x, t)$, $u = u_0(x) + u'(x, t)$, etc. and noting that the steady-state pressure is a constant, p_0, we have

$$\rho_0 u_0 = \text{const} = A, \qquad \rho_0 h_0 = \text{const} = [\gamma/(\gamma - 1)]p_0$$

and

$$Ah_{0,x} - (k/c_p)h_{0,xx} = Q_0/\pi R^2.$$

[13] Q is the heat input per unit time per unit distance in the flow direction.

From this, $h_0(x)$ can readily be obtained when Q_0 and $h(-\infty)$ are known. The perturbation terms in the sum: [Eq. (4.7) plus h times (4.6)] lead to the equation

$$\frac{\gamma}{\gamma - 1}(p_0 u_x' + p_t') - (k/c_p)h_{xx}' = Q'/\pi R^2, \tag{4.8}$$

where the primes denote the fluctuating components.

It follows that

$$u' - u_1' = \frac{\gamma - 1}{\gamma p_0}\left[\frac{k}{c_p}h_x' + (\pi R^2)^{-1}\int_0^X Q'\,dx - \frac{\gamma}{\gamma - 1}p_t' x\right]. \tag{4.9}$$

Thus, $u_2' - u_1'$ is given by Eq. (4.5) only when X is large enough so that h_x' vanishes to our order of accuracy. That is, $kh_x'/c_p p_0 \ll u_1'$. There are, of course, two contributions to h_x' in the downstream region; that associated with the acoustic wave and that of the entropy wave. If the contribution of each wave to each side of this inequality is estimated, it is seen that only when the entropy wave has decayed essentially to extinction (this occurs in a very small fraction of an acoustic wave wave length) does the inequality hold. Thus, we must choose X downstream of the heater at such a distance that only the "isentropic" waves are still of appreciable magnitude and then Eqs. (4.6) and (4.7) constitute the boundary conditions on the upstream and downstream acoustic waves. Having determined the acoustic waves, one could, if it were of interest, use Eqs. (4.6) and (4.9) to find u' just at the rear of the heater and use this value together with u_1' to find the contribution of the entropy wave $\varphi^{(3)}$. With $\varphi^{(3)}$ so determined, the decaying density, temperature, and velocity, fields behind the heater are known.

Equations (4.4) and (4.5), applied to the contributions of $\varphi^{(1)}$ and $\varphi^{(2)}$ at $x = \sigma$ (to obtain algebraic simplicity without essential loss of accuracy), lead to the characteristic equation

$$\begin{vmatrix} \sin[k_1'(1-\sigma)] - iz_1 \exp[-ik_1'(1-\sigma)], & -\nu[\sin k_2'\sigma - iz_2 \exp(-ik_2'\sigma)] \\ \cos[k_1'(1-\sigma)] - z_1 \exp[-ik_1'(1-\sigma)], & (1+\lambda)[\cos k_2'\sigma - z_2 \exp(-ik_2'\sigma)] \end{vmatrix} = 0 \tag{4.10}$$

where

$$z_1 = 1 - |N_1| \exp[-k_1''(1-\sigma)], \qquad z_2 = 1 - |N_2| \exp(-k_2''\sigma)$$

This is a rather messy transcendental equation for α but it can be treated by noting that λ and z_i are small compared to unity (but larger than the contributions we have consistently omitted). For the case $\lambda = z_i = 0$, the characteristic equation contains only real contributions (for real α) and has a real eigenvalue, α. If $k_2 = k_1$, it reduces to $\sin k_1 = 0$ and the solution of major interest is $k_1 = \pi$. It is easy to compute (numerically) the eigenvalues for $a_2/a_1 = k_1/k_2 \neq 1$, and such a solution can be used as the basis of a perturbation calculation for α. Denoting the zero order values of k_1' and k_2' by k_{10} and k_{20} we let $k_i' = k_{i0} + k_{i1}$. The characteristic equation can now be linearized in the k_{i1} noting that $k_{11}/k_{21} = a_1/a_2 = \nu$, and the resulting equation has in general, a complex solution.

If we denote by μ_1, the imaginary part of k_{11}, we obtain

$$\begin{aligned}[(1-\sigma) + \nu^2\sigma)] \cos[k_{20}\sigma] \cos[k_{10}(1-\sigma)] - \nu \sin[k_{20}\sigma] \sin[k_{10}(1-\sigma)]\mu_1 \\ = -\mathcal{I}m(\lambda) \sin[k_{10}(1-\sigma)] \cos[k_{20}\sigma] - (z_2 + z_1) \sin[k_{10}(1-\sigma)] \sin[k_{20}\sigma] \\ - (z_1 + z_2) \cos[k_{10}(1-\sigma)] \cos[k_{20}\sigma],\end{aligned}$$

and a negative value of μ_1 implies a negative imaginary contribution to α. Thus, the tube will sing when $\mu_1 < 0$. It is consistent, of course, that $\mathscr{I}m\ (\lambda)$ is always negative and the z_1 are positive functions of real α. μ_1 will be less than zero for the lowest mode only when the bracket multiplying it is negative. This corresponds to having the heater in a prescribed lower portion of the tube which would be $\sigma < 1/2$ for $a_1 = a_2$ but which is $\sigma < K$ where $K < 1/2$ for $\alpha_1 > \alpha_2$. Together with this position requirement, a negative μ_1 also requires that the λ contribution to the right side of the equation be larger than the other terms. It is not a desirable task to tabulate the dependence of this criterion on the many parameters involved, but the relation of certain predictions to the observed facts can readily be discussed. For a velocity, u_0, of 1.7 ft/sec and a heater temperature of the order of 800°F, the prediction of the foregoing analysis is that $\mathscr{I}m\ (\lambda)$ is large enough so that the tube should sing for ribbon breadths of .08 cm or greater, but that a wire of less than this size should require a higher temperature. These facts were verified by experiments conducted by J. J. Bailey at the Harvard University Combustion Laboratory. He observed that ribbons of breadths 3/16″, 3/32″, 1/4″, etc., play well at such a temperature (in an apparatus with $L = 75$ cm, $l = 90$ cm, $R = 5$ cm), but that a 1/16″ wire needed such a large temperature that several wires were burned out during the experiments.

Many investigators (see the bibliography) have noted that the tube will sing at the fundamental only when $\sigma < L'/2$ (L' denotes a correction for temperature). Bailey, however, has successfully "played" the second harmonic by placing the ribbon in the range $1/2 < \sigma < 3/4$. The eigenvalues of the characteristic equation predict a growing wave at essentially the second harmonic frequency under these circumstances.

Another experimental fact not reported by other investigators is the following. For a given heater position and various ribbon temperatures (and, hence, various sound levels), the ratio of intensity of second and first harmonics has been recorded. This ratio increases (monotonically and in an experimentally repeatable manner) with increase in sound level. This is consistent with the qualitative discussion of the large amplitude heater response at the end of Sec. 3. There, it was noted that an increasing fraction of the heat release induced by the fluctuating velocity could be expected to appear at frequency 2ω as w/u_0 increased. This is indeed the case and may well be the major influence in deciding the sound level of the tube. The conjecture is this: as the sound level increases, more and more of the energy associated with $\mathscr{I}m\ \lambda$ is stolen by the second harmonic until the first harmonic contribution of $\mathscr{I}m\ (\lambda)$ is balanced by the z_i. This is a crude picture which could readily be made more precise by a detailed knowledge of the large w/u_0 heater response.

We have not presented a quantitative account of the Harvard experiments here since the instrumentation was carried only to an accuracy consistent with the foregoing description. However, this singing tube analysis would not have evolved to the present stage without the enthusiastic experimental accompaniment of J. J. Bailey.

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—NOTES—

THE POST BUCKLING BEHAVIOR OF A CLAMPED CIRCULAR PLATE*

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1. Introduction. The von Kármán equations for large deflections of elastic plates have been completely solved by K. O. Friedrichs and J. J. Stoker for the case of a circular plate simply supported around the circumference where it is subjected to a uniform radial thrust in the plane of the plate, Refs. [1] and [2]. Their solution describes the plate behavior from the initial buckled state to the condition of the ratio of the edge thrust to initial buckling load becoming infinitely large.

The purpose of this note is to apply the methods developed in [1] to determine the post buckling behavior of a clamped circular plate. A modification of one of the methods is made which reduces the computation necessary to analyze certain aspects of the plate behavior.

2. Mathematical formulation. For assumed radially symmetric deformations, the von Kármán equations for a circular plate are a pair of non-linear ordinary differential equations, each of second order. These equations are

$$\nabla^4\phi = w_{,rr}(1/r)w_{,r}\ , \tag{1}$$

$$(\gamma h)^2\nabla^4 w + (1/r)\phi_{,r}w_{,rr} + (1/r)\phi_{,rr}w_{,r} = 0, \tag{2}$$

where ϕ is the stress function of the membrane stresses, w is the deflection of the middle surface of the plate, r is the radial coordinate from the plate center, h is the plate thickness and $\gamma^2 = 1/12(1 - \nu^2)$ where ν is Poisson's ratio. The radius of the plate is R and the quantities obtained by differentiating ϕ are stresses per modulus of elasticity, E. Equations (1) and (2) can be simplified by introducing new dependent variables p and q where

$$p = (1/r)\phi_{,r}\ , \text{ the compressive radial membrane stress divided by Young's modulus,} \tag{3}$$

$$q = -(R/r)w_{,r}\ . \tag{4}$$

Equations (1) and (2) then become

$$\{r^2[Gp - (1/2)q^2]\}_{,r} = 0, \tag{5}$$

$$\{r^2[\eta^2 Gq + pq]\}_{,r} = 0, \tag{6}$$

where

$$Gq \equiv (r^3 q_{,r})_{,r}(R^2/r^3) \tag{7}$$

and

$$\eta^2 = \gamma^2 h^2/R^2. \tag{8}$$

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The integration of (5) and (6) yields

$$Gp - (1/2)q^2 = 0, \tag{9}$$

$$\eta^2 Gq + pq = 0. \tag{10}$$

The constants arising from the integration are zero by continuity considerations. Equations (9) and (10) with appropriate boundary conditions completely define the problem. The boundary conditions at the edge of a clamped plate subjected to a radial edge thrust p^* are

$$p = p^* \quad \text{at} \quad r = R, \tag{11}$$

$$q = 0 \quad \text{at} \quad r = R. \tag{12}$$

The two other boundary conditions needed for a complete solution are at the center of the plate and are due to symmetry requirements. They are

$$p_{,r} = 0 \quad \text{at} \quad r = 0, \tag{13}$$

$$q_{,r} = 0 \quad \text{at} \quad r = 0. \tag{14}$$

3. Methods of solution. The boundary value problem considered here depends essentially upon one parameter N: the ratio of the edge pressure p^* to the lowest critical pressure p_{cr}. In [1] and [2] a perturbation method is used to solve (9) and (10) for low values of N, $1 \leq N \leq 2.5$. This perturbation method consists of expanding p, q and p^* as power series in a quantity ϵ which vanishes at the onset of buckling, i.e. at $N = 1$. The value of p_{cr} is obtained as one of the steps in the perturbation method and was found to be

$$p_{cr} = 14.68\eta^2.$$

This method becomes increasingly tedious as N increases and it is convenient to use a power series method for $N > 2.5$.

The power series method is mathematically simpler than the perturbation method but necessitates an estimate of the membrane stress and the curvature at the center of the plate. This estimate can be obtained from the results of the application of the perturbation method in the region of lower N. The power series method can be applied, however, without previous use of the perturbation method by utilizing an important aspect of the plate behavior.

One of the notable results of [1] and [2] was that at large values of N the membrane stresses become tensions in the interior of the simply supported plate and change abruptly to compressions in a narrow "boundary layer" at the plate edge. Since the same general behavior can be expected to occur for a clamped plate, the power series method can be modified to obtain the value of N at which the radial membrane stress at the center of the plate, p_0, changes from compression to tension. The solution of this problem can serve to give the estimate of the physical quantities needed for the further direct application of the power series method.

An asymptotic solution is used to determine the limit situation as N becomes infinitely large. From this solution the limiting value of the membrane stress in the interior of the plate can be determined.

4. Power series method. The introduction of a new independent variable α and new dependent variables π and κ into (9) and (10) permits a solution of those equations by power series expansions.

Hence, if

$$\left.\begin{aligned} \alpha &= Ar/R, & 0 \le \alpha \le A, \\ \pi &= p/A^2\eta^2, & \kappa = q/A^2\eta, \end{aligned}\right\} \tag{15}$$

where A is an arbitrary parameter, (9) and (10) become

$$(1/\alpha^3)(\alpha^3\pi_{,\alpha})_{,\alpha} = (1/2)\kappa^2, \tag{16}$$

$$(1/\alpha^3)(\alpha^3\kappa_{,\alpha})_{,\alpha} + \pi\kappa = 0, \tag{17}$$

with boundary conditions

$$\pi = \pi^* = p^*/\eta^2 A^2 \quad \text{at} \quad \alpha = A, \tag{18}$$

$$\kappa = 0 \quad \text{at} \quad \alpha = A, \tag{19}$$

$$\pi_{,\alpha} = 0 \quad \text{at} \quad \alpha = 0, \tag{20}$$

$$\kappa_{,\alpha} = 0 \quad \text{at} \quad \alpha = 0. \tag{21}$$

The variables π and κ can be represented in the form

$$\pi = \sum_{k=0}^{\infty} \pi_k \alpha^{2k}, \tag{22}$$

$$\kappa = \sum_{k=0}^{\infty} \kappa_k \alpha^{2k}. \tag{23}$$

The first term in each of these expansions is a function of physical quantities at the center of the plate. The remaining coefficients of each series can be expressed as functions of the first terms, π_0 and κ_0, by means of the recursion relations obtained from substituting (22) and (23) into the differential equations (16) and (17). These recursion relations are

$$2k(2k+2)\pi_k = (1/2) \sum_{m+n=k-1} \kappa_m \kappa_n , \tag{24}$$

$$2k(2k+2)\kappa_k = - \sum_{m+n=k-1} \pi_m \kappa_n . \tag{25}$$

The power series method of [1] and [2] consists of prescribing π_0 and κ_0 and solving for A and π^* from the equations obtained from boundary conditions (19) and (18). The corresponding values of p, q, p^* and N can then be calculated from the previous formulas.

The value of A is the lowest root of the equation

$$\sum_{k=0}^{\infty} \kappa_k A^{2k} = 0, \tag{26}$$

which is boundary condition (19) at the plate edge. This equation can be solved for A with a minimum of labor if $A \sim 1$. For $A = 1$, $\pi_0 = p_0/\eta^2$ and $\kappa_0 = q_0/\eta$ where p_0 and q_0 are the values of p and q at the center of the plate. These values can be estimated for a particular N from the results obtained by the application of the perturbation method for lower values of N.

From the results of [1] and [2] it is reasonable to expect that, for the case of a clamped plate, an N exists for which $p_0 = 0$. Therefore, for this special case, $\pi_0 = 0$ and all the coefficients in the expansions (22) and (23) can be written in terms of one parameter, κ_0. The value of κ_0 can be determined from the solution of (26) in which A can be prescribed

to be unity. That is, κ_0 is the lowest root of

$$\sum_{k=0}^{\infty} \kappa_k = 0, \tag{27}$$

which was found to be $\kappa_0 = 27.11$ using coefficients up to κ_{20}. The corresponding value of π^* was found from (22) and (18) to be 29.11. The value of N for the condition $p_0 = 0$ was found from the relations $N = p^*/p_{cr} = \pi^*/14.68$ to be 1.98.

With this information the first section of the curve for the variation of the radial membrane stress at the plate center, p_0, with N can be sketched as shown in Fig. 1.

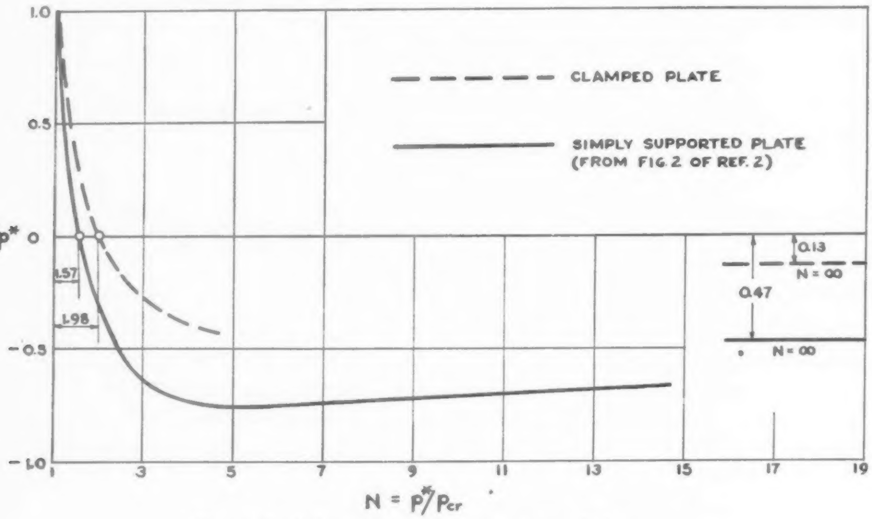

FIG. 1. Radial membrane stress at center of plate.

A similar curve could be drawn for q_0. The general behavior of these curves, from physical considerations, should not differ from those obtained in [2] for simply supported plates. These curves can then serve to provide estimates of p_0 and q_0 for use in the further application of the power series method. When the limiting case of $N \to \infty$ is solved, an estimate of p_0 and q_0 can be obtained for the complete range of N, (Fig. 1). In this manner the power series method can be conveniently used for the range $1 \leq N \leq 15$, thereby being independent of the results of the perturbation method. For N about equal to and greater than 15, the power series method becomes unwieldly and an asymptotic development is preferable.

5. Asymptotic solution. The solution of (9) and (10) for N tending to infinity for the clamped plate follows, in general, the method described in [1] for the simply supported plate. The computation necessary for the clamped plate was greater than that required for the simply supported plate. This solution shows that as $N \to \infty$ the interior of the plate acts as a membrane subjected to a uniform tensile stress equal to .131 times the compressive thrust at the plate edge. That is,

$$\lim_{N \to \infty} p = -.131p^*.$$

This is also the asymptotic value of p_0 and is indicated on the graph, (Fig. 1).

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AN APPROXIMATE SOLUTION TO THE NAVIER-STOKES EQUATIONS*

By MORTON MITCHNER (*Harvard University*)

The purpose of this note is to show how a new approximate solution of the Navier-Stokes equations may be constructed from any given exact solution having a certain specified form. We shall suppose that we are given an exact solution of the Navier-Stokes equations for an incompressible viscous fluid having a velocity field $\mathbf{q}' = (q_1', q_2', q_3') = (u', v', w')$ specified in the form

$$u' = \alpha U(y, z, t),$$
$$v' = \alpha V(y, z, t), \tag{1}$$
$$w' = \alpha W(y, z, t).$$

U, V, and W denote three functions of the position vector $\mathbf{r} = (x_1, x_2, x_3) = (x, y, z)$, and the time coordinate t; α denotes a dimensionless constant. For consistency with the equations of motion (upon taking the divergence of the Navier-Stokes equation, and employing the continuity condition), the pressure p' (and density ρ) must satisfy

$$-\frac{1}{\rho}\nabla^2 p' = \sum_{i,k} \frac{\partial q_i'}{\partial x_k}\frac{\partial q_k'}{\partial x_i}.$$

Hence, it is sufficient to assume that the pressure field has the form

$$p' = \alpha^2 P(y, z, t). \tag{2}$$

Assuming the existence of the above exact solution, we can construct a new approximate solution $[\mathbf{q} = (u, v, w), p]$ of the equations of motion for an incompressible viscous fluid, and this solution is given by

$$\begin{aligned} u &= u_0(a + by) + u' - (t - t_0)bu_0v', \\ v &= v', \\ w &= w', \\ p &= \text{constant}, \end{aligned} \tag{3}$$

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(u_0, a, b, and t_0 are constants). This solution is valid to first order in α for $t \geqq t_0$ under the hypotheses that

$$\left.\begin{array}{ll} \text{(a)} & \text{the spatial derivatives of } \mathbf{q}' \text{ are bounded for } t \geqq t_0 \\ \text{(b)} & \alpha \ll 1, \\ \text{(c)} & \alpha\, bu_0\,(t - t_0) \ll 1. \end{array}\right\} \qquad (4)$$

The velocity field $\mathbf{q}(y, z, t)$ may be regarded as the subsequent time development of an initial disturbance $\mathbf{q}'(y, z, t_0)$ superimposed upon a Couette-type shear flow at time t_0.

The validity of the preceding statement may be checked by direct substitution into the Navier-Stokes equations. Thus, for the u component

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2 u$$

and hence, employing (3),

$$\frac{\partial u'}{\partial t} + v'\frac{\partial u'}{\partial y} + w'\frac{\partial u'}{\partial z} - bu_0(t - t_0)\left[\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial y} + w'\frac{\partial v'}{\partial z}\right]$$
$$= \nu\nabla^2 u' - bu_0(t - t_0)\nu\nabla^2 v'. \qquad (5)$$

In virtue of the fact that u' and y' are exact solutions of the Navier-Stokes equations, Eq. (5) states that

$$bu_0(t - t_0)\frac{1}{\rho}\frac{\partial p'}{\partial y} = 0. \qquad (6)$$

But $p' = \alpha^2 P(y, z, t)$ and for $t \geqq t_0$, $\partial P/\partial y$ is bounded. The term on the left side of Eq. (6) will therefore be of second order in α provided t also satisfies the condition

$$\alpha^2 bu_0(t - t_0) \ll \alpha, \qquad \text{or} \qquad \alpha bu_0(t - t_0) \ll 1.$$

In a similar fashion, it may also be shown by direct substitution that v and w satisfy the Navier-Stokes equations.

A particular example of the above general result is provided by the known velocity field describing the decay and diffusion of an infinitely long vortex filament initially concentrated on the x axis.

$$\begin{aligned} u' &= 0, \\ v' &= -\frac{K}{2\pi} z \frac{[1 - \exp(-r^2/4\nu t)]}{r^2}, \\ w' &= \frac{K}{2\pi} y \frac{[1 - \exp(-r^2/4\nu t)]}{r^2}, \qquad r^2 = y^2 + z^2. \end{aligned} \qquad (7)$$

K denotes the initial circulation or strength of the vortex filament, and is to be associated in its dimensionless form, Kb/u_0, with the parameter of smallness α. For any $t_0 \neq 0$, the solution (7) satisfies the hypotheses (4), and consequently Eqs. (3) and (7) describe the behavior [for $t - t_0 \geqq 0$, but $(t - t_0) \ll 1/bu_0\alpha$] of a vortex filament superimposed

on a Couette-type shear flow, the vortex filament being aligned with the direction of flow.

The particular solution stated above exhibits some interesting properties as regards the exchange of vorticity between components. We note that

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{\partial w'}{\partial y} - \frac{\partial v'}{\partial z} = \xi',$$

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \frac{\partial u}{\partial z} = -(t - t_0)bu_0 \frac{\partial v'}{\partial z},$$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\frac{\partial u}{\partial y} = -u_0 b + (t - t_0)bu_0 \frac{\partial v'}{\partial y}.$$

The vorticity associated with the vortex filament disturbance, ξ', remains completely unaffected by the presence of the shear flow and proceeds to decay as if the shear flow were absent. However, the η component of the vorticity, initially zero, begins to grow at the expense of the vorticity of the shear flow. The action of the ξ component is thus analogous to that of a chemical catalyst; while ξ itself remains unaffected by the shear flow, it causes a production of η, drawing upon the infinite field of oriented vorticity in the shear flow. It may be shown that this phenomenon (that ξ is unaffected by the presence of the shear flow) is actually independent of the specific form of the shear flow.

*APPENDIX**

In connection with the preceding remarks, the editors of the Quarterly have brought to the author's attention an investigation by Berker.** Although there is no direct connection between the present work and that of Berker, there does exist a superficial similarity which may be worthy of clarification.

Berker assumes that an exact solution of the Navier-Stokes equations for an incompressible viscous fluid with respect to an inertial frame of reference *oxyz* is provided by the velocity field $\mathbf{q}'(x, y, z, t)$. Using this given vector point function, Berker then defines a vector field $\mathbf{q}_{r}(X, Y, Z, t) = \mathbf{q}'(X, Y, Z, t)$ with respect to a moving frame of reference *OXYZ*. Corresponding to the motion $\mathbf{q}_{r}(X, Y, Z, t)$ with respect to *OXYZ*, there will exist a motion $\mathbf{q}_B(x, y, z, t)$ with respect to *oxyz*. Berker then determines the conditions under which $\mathbf{q}_B(x, y, z, t)$ will be an exact solution of the Navier-Stokes equations for an incompressible viscous fluid (his Eq. 6.14) and indicates the construction of $\mathbf{q}_B(x, y, z, t)$ in terms of $\mathbf{q}'(x, y, z, t)$ (his Eq. 6.13).

For the particular form of the assumed initial exact solution provided by Eq. (1), Berker's new exact solution has the form

$$\begin{aligned} u_B(y, z, t) &= a^{\cdot}(t) + u'(Y, Z, t), \\ v_B(y, z, t) &= b^{\cdot}(t) - (z - c)\Omega + v'(Y, Z, t) \cos \Omega t - w'(Y, Z, t) \sin \Omega t, \\ w_B(y, z, t) &= c^{\cdot}(t) + (y - b)\Omega + v'(Y, Z, t) \sin \Omega t + w'(Y, Z, t) \cos \Omega t, \end{aligned} \tag{8}$$

*Received Jan. 7, 1954.

**A. R. Berker, *Sur quelques cas d'intégration des équations du mouvement d'un fluide visqueux incompressible*, Institut de Mécanique des Fluides de L'Université de Lille, 1936.

where Ω is a constant, where $a(t)$, $b(t)$, $c(t)$ are arbitrary functions of t, and where

$$Y = (y - b) \cos \Omega t + (z - c) \sin \Omega t,$$

$$Z = -(y - b) \sin \Omega t + (z - c) \cos \Omega t.$$

Comparison of (3) with (8) makes quite evident that these two solutions are essentially different. Whereas (8) is exact, (3) is approximate. Furthermore, (3) cannot be derived as an approximation from (8).

A MEASURE OF THE AREA OF A HOMOGENEOUS RANDOM SURFACE IN SPACE*

By STANLEY CORRSIN (*Aeronautics Department, The Johns Hopkins University*)

Introduction. We are given an indefinitely large space containing random surface or surfaces homogeneously located in the mean. The problem is to relate the average area of surface per unit volume of space to a simpler statistical quantity, in particular the average number of cuts per unit length made by a straight randomly directed sampling line with the surface.

The plane case will be studied first. After the three dimensional case, illustrative application will be made to the problem of extending to two and three dimensional variables a theorem of S. O. Rice on the average rate of occurrence of any particular value of a one dimensional random variable. Possible use in describing fluid mixing is also indicated.

Two dimensions. Given a plane "homogeneously" inscribed with contour or contours of arbitrary shape. The homogeneity is statistical, i.e. any statistical function associated with the contours is invariant to a translation of coordinate system in the plane. Let $\mathfrak{L}$ be the average contour length enclosed in unit area and let n be the average number of cuts per unit length made by an arbitrary straight traverse line crossing the plane. For a non-isotropic field n is averaged over all traverse directions with uniform weighting; for an isotropic field, any single line will do.

Draw a "very large" square in the plane, L on a side, and subdivide it into "very narrow" traverse strips parallel to one pair of sides.

"Very large" here denotes L so large that averages over L or L^2 are satisfactorily close to their asymptotic values. For example it requires that each traverse strip cross the contours a very large number of times and that the length of contour in L^2 divided by L^2 be as close as we like to $\mathfrak{L}$. "Very narrow" denotes δ so small that virtually all of the intercepted contour segments in a strip can be approximated by secants. This gives restriction on the permissible number of corners and contour intersections.

We imagine the square and strip structure rotated through 180° for averaging purposes in case the field is not isotropic. Then the average number of crossings in one

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strip is nL. The number of strips is L/δ. Hence the expected total contour length in the square is

$$\mathfrak{L}_{L^2} = \frac{nL^2}{\delta} \langle l \rangle, \tag{1}$$

where $\langle l \rangle$ is the average contour segment per crossing, so (1) can be written

$$\mathfrak{L} = \frac{n}{\delta} \langle l \rangle. \tag{2}$$

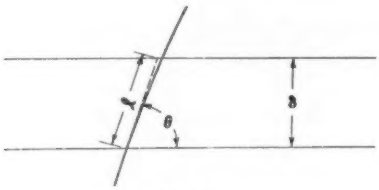

FIG. 1

To obtain $\langle l \rangle$, consider a typical crossing (Fig. 1). For any one crossing

$$l = \frac{\delta}{|\sin \theta|} \cdot \tag{3}$$

However, for each traverse orientation, the probability of intersection at angle θ with a contour element is proportional to the projected length of the element on a line perpendicular to the traverse direction, i.e. to $|\sin \theta|$. With normalization this becomes the probability density of

$$\beta_\theta(\theta) = \begin{cases} \frac{1}{2} \, |\sin \theta| & -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\ 0 & \text{elsewhere.} \end{cases} \tag{4}$$

But

$$\langle l \rangle = \int_0^\infty l \beta_l(l) \, dl = 2 \int_0^{\pi/2} l(\theta) \beta_\theta(\theta) \, d\theta. \tag{5}$$

Substituting (4) into (5) we find

$$\langle l \rangle = \frac{\pi}{2} \, \delta, \tag{6}$$

whence

$$n = \frac{2}{\pi} \, \mathfrak{L}. \tag{7)*$$

This result is easily seen to be consistent with the solution of Buffon's "needle problem"†

*Arrived at independently by P. V. Danckwerts of Cambridge University (private communication).

†See, for example, Uspensky: *Introduction to mathematical probability*, McGraw-Hill Book Co., 1937.

in the special case when the contours are equidistant parallel straight lines.

For an isotropic field $\mathcal{L}$ can thus be determined by the intersection rate along a single sampling traverse.

Three dimensions. Now we have a three dimensional space homogeneously inscribed with surface or surfaces of arbitrary shape. Define $\mathcal{A}$ as the average surface area per unit volume of space and n as the average number of cuts per unit length made by a random straight line traverse across the space. For a nonisotropic field n is averaged over all directions with equal weight; for an isotropic field a single traverse suffices.

Take a "very large" cube (L on a side) and subdivide into "very narrow" square traverse tubes ($\delta \times \delta \times L$) with all faces parallel to those of the cube.

The average number of surface crossings in a single tube is nL. The number of tubes in L^3 is L^2/δ^2. Hence the expected surface area in the cube is

$$\mathcal{A}_{L^3} = nL \frac{L^2}{\delta^2} \langle a \rangle, \tag{8}$$

where $\langle a \rangle$ is the average area segment per crossing. $\mathcal{A}_{L^3} = \mathcal{A} \cdot L^3$, so (8) can be written

$$\mathcal{A} = \frac{n}{\delta^2} \langle a \rangle. \tag{9}$$

To obtain $\langle a \rangle$, consider a typical crossing (Fig. 2), with φ the angle between tube axis and normal to surface element.

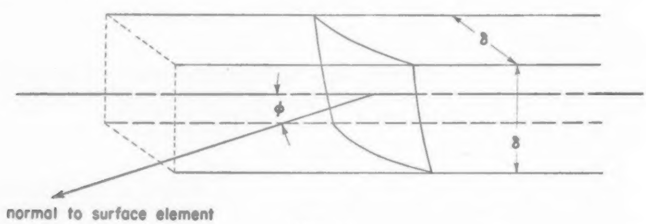

FIG. 2

$$a = \frac{\delta^2}{|\cos \varphi|}. \tag{10}$$

For each traverse orientation (i.e. cube orientation) the probability of intersection at angle φ with a surface element is proportional to the projected area of the surface on a plane perpendicular to the tube axis, i.e. to $|\cos \varphi|$ times a measure of relative solid angle giving φ, i.e. $|\sin \varphi|$. With normalization constant, this gives the probability density of φ

$$\beta_\varphi(\varphi) = \begin{cases} 2 \, |\cos \varphi| \cdot |\sin \varphi| & 0 \leq \varphi \leq \frac{\pi}{2} \\ 0 & \text{elsewhere;} \end{cases} \tag{11}$$

therefore

$$\langle a \rangle = \int_0^\infty a \beta_a(a) \, da = \int_0^{\pi/2} a(\varphi) \beta_\varphi(\varphi) \, d\varphi. \tag{12}$$

Substituting (10) and (11) into (12), we find

$$\langle a \rangle = 2\delta^2. \tag{13}$$

Whence

$$n = \frac{\mathcal{A}}{2}. \tag{14}$$

Again, for an isotropic field, $\mathcal{A}$ can be determined by the intersection rate along a single sampling traverse.

Application I: Rice's theorem in two and three dimensions.

Given a two dimensional stationary ($\equiv$ homogeneous) isotropic random variable $u(x, y)$. The iso-value lines corresponding to $u = u_c$ will be an isotropic field of contours in the plane, and we can seek the average occurrence length per unit area of the value u_c :

$$\mathcal{L}_{u_c} = \frac{\pi}{2} n_{u_c} , \tag{15}$$

where n_{u_c} is the average number of u_c occurrences per unit length along the linear sampling traverse in any direction. The traverse yields a one dimensional random variable to which we can apply a theorem of Rice:*

$$n_{u_c} = \int_{-\infty}^{\infty} \gamma(u_c , u') \mid u' \mid du', \tag{16}$$

where $u' \equiv \partial u/\partial s$, the derivative in the direction of our sampling traverse and $\gamma(u, u')$ is the joint probability density of u and its slope in this sample. From (15) and (16)

$$\mathcal{L}_{u_c} = \frac{\pi}{2} \int_{-\infty}^{\infty} \gamma(u_c , u') \mid u' \mid du'. \tag{17}$$

In the special case of jointly Gaussian u and u', we use Rice's simplified result to obtain

$$\mathcal{L}_{u_c} = \frac{1}{4} \left\{ -\frac{\Psi''(0)}{\Psi(0)} \right\}^{1/2} e^{-u^2/2\Psi(0)} \tag{18}$$

where $\Psi(\sigma)$ is the auto-correlation, $\langle u(s)u(s + \sigma) \rangle$.

For three dimensions, the corresponding equations are

$$\mathcal{A}_{u_c} = 2 \int_{-\infty}^{\infty} \gamma(u_c , u') \mid u' \mid du' \tag{19}$$

and, for jointly Gaussian u and u',

$$\mathcal{A}_{u_c} = \frac{1}{\pi} \left\{ -\frac{\Psi''(0)}{\Psi(0)} \right\}^{1/2} e^{-u^2/2\Psi(0)}. \tag{20}$$

For non-isotropic, homogeneous $u(x, y)$, $\gamma(u, u')$ is a function of traverse direction, and we must include the operation of averaging over all directions.

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*S. O. Rice, *Mathematical analysis of random noise*, Bell System Tech. J., **23** (3) and **24** (1), July 1944 and January 1945.

Application II: Fluid mixing.

The average interfacial area per unit volume is a significant measure of the "degree of mixedness" in an isotropic field of two molecularly immiscible liquids. It is, however, inaccessible to straightforward experimental determination. Equation (14) permits its calculation from the simpler process of interfacial-intersection counting along a linear traverse through the mixture.

In the homogeneous mixing of two gases or molecularly miscible liquids (e.g. turbulent mixing) it is possible that the notion of interfacial area can be replaced either by the surface area on which the concentration fluctuation is zero, or by the surface on which the concentration gradient magnitude has a local maximum.

EQUILIBRIUM OF MEMBRANES ELASTICALLY SUPPORTED AT THE EDGES*

By V. G. HART (*Dublin Institute for Advanced Studies*)

Abstract. The problem considered is that of finding the statical deflection of a stretched membrane, subjected to a uniform pressure on one side and elastically supported at the edges. The deflection of the membrane is supposed small and the problem reduces to solving Laplace's equation with mixed boundary conditions. Solutions are given for the cases where the bounding curve of the membrane is (i) an equilateral triangle, and (ii) a rectangle.

1. Introduction. We consider the problem of finding the statical deflection of a membrane originally lying in a plane (the neutral plane), when subjected to a uniform pressure on one side, its edge being elastically supported. This means that the edge can move in a direction perpendicular to the neutral plane, but is restrained at any point by a force proportional to the deflection at that point. Small deflections only being considered, the tension is a constant, and the problem reduces to solving the boundary-value problem (3) for an edge of arbitrary shape; solutions are given for (i) a membrane in the form of an equilateral triangle and (ii) a rectangular membrane.

Imagine a membrane stretched to a uniform tension T and bounded by any plane curve B. A uniform pressure P now acts on one side of the membrane which takes up a statical deflected position with deflection w. At any point on the edge the tension gives a component of force per unit length in the direction perpendicular to the neutral plane of amount $-T\,\partial w/\partial n$, ($\partial n$ being the outward normal element to B in the neutral plane), and this is balanced by the elastic force of constraint, which we write w/k, where k is a constant.

The appropriate partial differential equation and boundary condition for w are therefore

$$\Delta w = -P/T \text{ in } S, \qquad (w)_B = -kT(\partial w/\partial n)_B\,, \tag{1}$$

S being the domain of the membrane.

We now make the transformation

$$u = \tfrac{1}{2}r^2 + 2Tw/P, \tag{2}$$

*Received Feb. 1, 1954.

where r is the distance from any fixed point in the plane of the problem, and from (1) we find that u must satisfy Laplace's equation with mixed boundary conditions, viz.,

$$\Delta u = 0 \text{ in } S, \qquad \left(u + c\frac{\partial u}{\partial n}\right)_B = \frac{1}{2}r^2 + c\frac{\partial}{\partial n}\left(\frac{1}{2}r^2\right), \tag{3}$$

where $c = kT$. Our procedure is to solve (3) for u in the case of the particular boundary B chosen, and then to find the deflection w from (2).

We note that on putting $c = 0$ in (3) the boundary-value problem reduces to the torsion problem, and this fact affords a check on the results obtained.

The Dirichlet integral of w, which is intimately connected with Laplace's equation, represents in this problem the excess potential energy stored in the deflected membrane. The total energy V_1 stored in the membrane is

$$V_1 = \frac{1}{2}T\int_S\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]dS + C, \tag{4}$$

where C is the energy stored in the undeflected membrane. On taking up its deflected position, work is also done on the elastic support and the potential energy V_2 contained in this is

$$V_2 = \frac{1}{2k}\int_B w^2\,db, \tag{5}$$

where db is an element of length of the boundary B. Using the boundary-condition on w in (1) and also Green's theorem we find

$$\begin{aligned} V_2 &= -\frac{1}{2}T\int_B w\frac{\partial w}{\partial n}\,db, \\ &= \frac{1}{2}P\int_S w\,dS - \frac{1}{2}T\int_S\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right]dS, \\ &= \frac{1}{2}P\int_S w\,dS - V_1 + C, \end{aligned} \tag{6}$$

since $\Delta w = -P/T$ in S. Thus the total potential energy V stored in membrane and support is

$$V = V_1 + V_2 = \frac{1}{2}P\int_S w\,dS + C, \tag{7}$$

as is indeed obvious.

If B is a circle, the solution of (3) is very simple, viz.,

$$u = \tfrac{1}{2}a^2 + ca, \tag{8}$$

where a is the radius of the circle, and the origin is taken at its centre. The corresponding deflection is

$$w = \frac{Pa^2}{4T}\left[1 - \frac{r^2}{a^2} + 2\frac{c}{a}\right]. \tag{9}$$

2. Membrane in the form of an equilateral triangle. We now consider the case where the membrane is an equilateral triangle of height $3a$. We take the origin and axes as in Fig. 1. To solve the boundary-value problem (3) for this domain, we seek a harmonic function possessing the symmetry properties of the triangle and also satisfying the boundary conditions. It is obvious that a function with one of the symmetries of the triangle—invariance under a rotation of axes through $2\pi/3$—will satisfy the boundary conditions on all three sides of the triangle if it satisfies them on one—say on $x = -a$. Bearing in mind the other symmetry of the triangle (reflection in $0x$), we accordingly choose

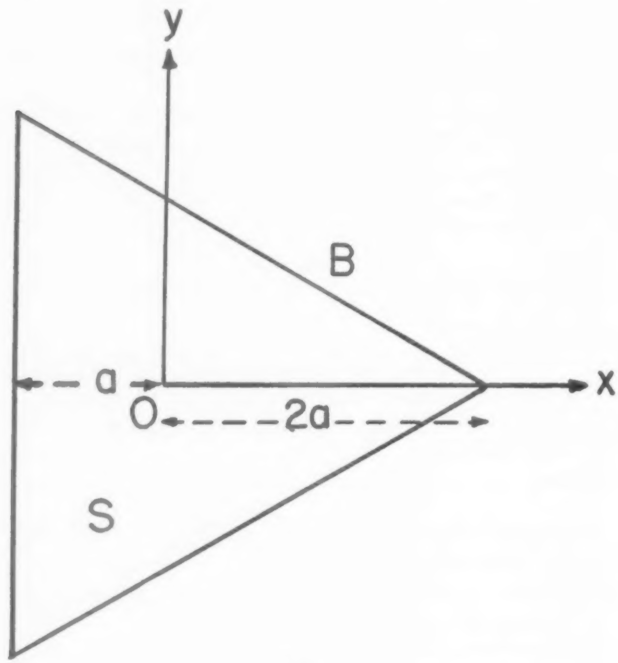

FIG. 1

$$u = A + B(z^3 + z^{*3}), \qquad z = x + iy, \qquad z^* = x - iy, \tag{10}$$

which is harmonic and has both the symmetries. We adjust the constants A and B to satisfy the boundary condition on $x = -a$, which reads:

$$\left(u - c\frac{\partial u}{\partial x}\right)_{x=-a} = \frac{1}{2}(a^2 + y^2) + ca. \tag{11}$$

On substituting from (10) we find A and B by comparing coefficients of y

$$A = \frac{a(2a^2 + 6ac + 3c^2)}{3(a + c)}, \qquad B = \frac{1}{12(a + c)}. \tag{12}$$

The problem (3) is now solved for u, and by (2) the deflection

$$w = \frac{Pa^2}{2T}\left[\frac{3c^2/a^2 + 6c/a + 2}{3(1 + c/a)} + \frac{x^3 - 3xy^2}{6a^3(1 + c/a)} - \frac{1}{2a^2}(x^2 + y^2)\right]. \tag{13}$$

The level lines are approximately circles for large values of the dimensionless parameter c/a, i.e. for a very weak elastic support on the edge.

The total potential energy stored in the membrane and support is by (7)

$$V = C + \frac{3^{3/2}P^2a^4}{40T}\left[\frac{10\,c^2/a^2 + 15\,c/a + 3}{1 + c/a}\right]. \tag{14}$$

3. Rectangular membrane. We consider now the case of a rectangular membrane (length $2a$, breadth $2b$). The axes are taken so that the sides have the equations $x = \pm a$, $y = \pm b$.

To solve the boundary-value problem (3), we introduce an auxiliary function v by the transformation

$$u = \tfrac{1}{2}(x^2 - y^2) + b^2 + 2cb + v, \tag{15}$$

and (3) becomes the following boundary-value problem for v

$$\Delta v = 0 \quad \text{in} \quad S, \qquad \begin{cases} \text{(A)} \quad v \pm c\,\dfrac{\partial v}{\partial y} = 0 & \text{on} \quad y = \pm b, \\ \text{(B)} \quad v \pm c\,\dfrac{\partial v}{\partial x} = y^2 - b^2 - 2cb & \text{on} \quad x = \pm a; \end{cases} \tag{16}$$

note the homogeneous boundary conditions on $y = \pm b$.

We propose to solve (16) for v and then find the deflection w through the transformations (15) and (2).

Since v is harmonic we take as the solution of (16)

$$v = \sum_{n=0}^{\infty} A_n \cosh(\lambda_n x)\cos(\lambda_n y), \tag{17}$$

where λ_n and A_n are to be determined. Application of the boundary condition (16A) gives the following equation for λ_n

$$\cot(b\lambda_n) = c\lambda_n\,. \tag{18}$$

This equation determines an infinite sequence of values. We shall use only the positive values, numbering them in order of increasing magnitude. For large values of n we have

$$\lambda_n \sim \frac{n\pi}{b} + \frac{1}{cn\pi}\,. \tag{19}$$

The functions $\cos \lambda_n y$ form an orthogonal set; direct calculation gives by use of (18):

$$\int_{-b}^{b} \cos(\lambda_m y)\cos(\lambda_n y)\,dy = \begin{cases} 0 & \text{for} \quad m \neq n, \\ b + c\sin^2(\lambda_n b) & \text{for} \quad m = n. \end{cases} \tag{20}$$

We expand the right-hand side of the boundary condition (16B) in terms of these orthogonal functions, writing

$$y^2 - b^2 - 2bc = \sum_{n=0}^{\infty} B_n \cos \lambda_n y, \qquad (-b \leq y \leq b); \tag{21}$$

the coefficients are found by using (20) and are

$$B_n = -4 \sin (\lambda_n b)/[\lambda_n^3(b + c \sin^2 \lambda_n b)]. \tag{22}$$

The boundary condition (16B) is now applied to v as defined by (17) and this requires that A_n should satisfy

$$y^2 - b^2 - 2cb = \sum_{n=0}^{\infty} A_n(\cosh \lambda_n a + c\lambda_n \sinh \lambda_n a) \cos \lambda_n y, \qquad (-b \leq y \leq b). \tag{23}$$

Comparing (23) with (21) we see that

$$A_n = B_n/(\cosh \lambda_n a + c\lambda_n \sinh \lambda_n a), \tag{24}$$

where B_n is defined by (22). The coefficients A_n in (17) are thus determined and so the solution v of (16) is found.

Using (15) and (2) we find the deflection to be

$$w = \frac{Pb^2}{2T}\left[1 + 2\frac{c}{b} - \left(\frac{y}{b}\right)^2 - 4\sum_{n=0}^{\infty} \frac{\sin^2 \lambda_n b}{(\lambda_n b)^3[1 + (\sin 2\lambda_n b/2\lambda_n b)](\tan \lambda_n b + \tanh \lambda_n a)} \cdot \frac{\cos (\lambda_n y) \cosh (\lambda_n x)}{\cos (\lambda_n b) \cosh (\lambda_n a)}\right], \tag{25}$$

where we have simplified the coefficients A_n in (24) by the use of (18).

As already remarked we get the torsion problem on putting $c = 0$; Eq. (25) is then easily seen to reduce to the known solution for the torsion of a beam of rectangular section, λ_n now being $(2n + 1)\pi/2b$.

Due to the presence of the term $\cosh (\lambda_n x)/\cosh (\lambda_n a)$ which has the asymptotic value $\exp [\lambda_n(|x| - a)]$ when n is large [λ_n being given by (19)], the series in (25) is uniformly and absolutely convergent in both x and y for the intervals $|x| < a$, $|y| < b$. Also if we differentiate twice with respect to x or y, the resulting series is still uniformly and absolutely convergent for the same reason. Thus the operation of term-by-term differentiation is legitimate, and so w, as in (25), is the solution of the boundary-value problem (1) for a rectangular boundary.

The potential energy stored in the membrane and support is

$$V = \frac{2P^2b^4}{3T}\left[\frac{a}{b}\left(1 + 3\frac{c}{b}\right) - 6\sum_{n=0}^{\infty} \frac{\sin^2 \lambda_n b}{(\lambda_n b)^5[1 + (\sin 2\lambda_n b/2\lambda_n b)]} \cdot \frac{\tan (\lambda_n b) \tanh (\lambda_n a)}{(\tan \lambda_n b + \tanh \lambda_n a)}\right] + C. \tag{26}$$

This statical problem for the rectangular membrane as discussed above is mathematically similar to a dynamical problem treated by Rayleigh (*Theory of sound*, Vol. I, pp. 200-204) viz., that of a vibrating string with ends elastically supported; this involves the equation (18) for the computation of proper frequencies in a certain limiting case.

In conclusion, I would like to express my gratitude to Professor J. L. Synge for his aid and criticism in the preparation of this paper.

THE SOLUTION OF EIGENVALUE PROBLEMS OF INTEGRAL EQUATIONS BY POWER SERIES*

BY J. R. M. RADOK (*Dept. of Supply, Aeronautical Research Laboratories, Melbourne, Australia*)

Summary. The characteristic equations and eigensolutions of Fredholm integral equations of the second kind with symmetrical kernels of the homogeneous polynomial type are obtained by use of power series. The method is applied to the particular case of the kernel $| x - y |$ and it is indicated that it may readily be extended to more general types of equations.

1. Introduction. One of the standard methods of solution of ordinary linear differential equations makes use of power series. Although certain types of integral equations correspond to such differential equations, to the author's knowledge no equivalent method for the direct solution of these equations has been developed up to date. It is the object of this report to fill this gap.

In [1] a method of solution of Fredholm integral equations of the second kind with symmetrical kernels of the polynomial type has been developed by Heller and Radok. This method was conceived as an iteration process. The only difference from the normal iteration procedure arose from the fact that the initial function, which in the commonly used process is arbitrary, was determined by the kernel of the integral equation. The choice of the initial function was thus based on the observation that the integral operator in the equation transformed an arbitrary power of the integration variable into certain "dominant" terms with exponents independent of the exponent of the power originally substituted and into one term of higher order. In other words, if the integral operator is given by

$$L\varphi = \int_0^1 K(x, y)\varphi(y)\,dy, \tag{1.1}$$

then

$$L(y^m) = x^{m+k}c_m + \sum_{l=\mu}^{\nu} A_{ml}x^l, \tag{1.2}$$

where μ, ν, k only depend on the kernel. Substituting for the initial function φ_0 a polynomial with arbitrary coefficients of the type represented by the "dominant" term in (1.2), applying to this initial function an arbitrary number of iterations and comparing then coefficients of powers of x on both sides of the integral equation

$$\varphi = \lambda L(\varphi), \tag{1.3}$$

the corresponding number of exact terms of the power expansions of the eigensolutions and of the characteristic equation may be obtained.

In this report the above process, which still contains formally some of the disadvantages of iteration processes, will be generalized into a straightforward power series method. In order to simplify the analysis, consideration will here be restricted to kernels of the homogeneous polynomial type. However it should be stressed that the method is easily extended to much wider classes of kernels. In fact, it will become obvious that

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it is applicable to kernels involving a finite number of linearly independent functions, in which case the desired eigensolutions have to be expanded in infinite series of the complete system of functions occurring in the kernel. In particular, this will be the case for kernels involving trigonometric functions, when the eigensolutions will be obtained in the form of Fourier series.

The general method of solution will be developed in Sec. 2, while one of the simplest kernels of the special type under consideration will be investigated in Sec. 3, in order to demonstrate the ease with which the method may be applied to actual problems. In Ref. [1], two more complicated types of kernels have been studied for which the exact solutions are known and which arise from the problems of natural vibrations of a uniform beam and of a wedge, clamped at one end.

2. The general method. As indicated in the introduction, consideration will be restricted to Fredholm integral equations of the second kind with symmetrical kernels of the homogeneous polynomial type, i.e., to equations of the type:

$$\varphi(x) = \lambda L(\varphi) = \lambda\left[\int_0^x P_{n-1}(x, y)\varphi(y)\,dy + \int_x^1 P_{n-1}(y, x)\varphi(y)\,dy\right], \tag{2.1}$$

where

$$P_{n-1}(x, y) = \sum_{(i,k)}^{n-1} A_{ik}x^i y^k, \qquad i + k = n - 1. \tag{2.2}$$

If it is assumed that $\varphi(x)$ is continuous and differentiable, it is easily seen that φ satisfies a differential equation of the form

$$\varphi^{(n)} = \lambda \sum_{j=0}^{n-1} p_j \varphi^{(j)}, \tag{a}$$

where p_j are polynomials in x; hence the solutions φ must be integral functions, i.e.,

$$\varphi(x) = \sum_{l=0}^{\infty} a_l x^l, \tag{2.3}$$

and the use of power series is justified.

Consider the operator

$$L(\varphi) = \int_0^x P_{n-1}(x, y)\varphi(y)\,dy + \int_x^1 P_{n-1}(y, x)\varphi(y)\,dy. \tag{2.4}$$

In particular,

$$L(y^l) = c_l x^{n+l} + \sum_{k=0}^{n-1} x^k A_{kl}^* , \tag{2.5}$$

where

$$c_l = \sum_{(i,k)=0}^{n-1} \frac{(i - k)A_{ik}}{(l + i + 1)(l + k + 1)} = O\left(\left(\frac{1}{l+1}\right)^2\right), \qquad l = 0, 1, \cdots \tag{2.6}$$

$$A_{kl}^* = \sum_{i=0}^{n-1} \frac{A_{ik}}{l + i + 1} = O\left(\frac{1}{l+1}\right), \qquad \begin{array}{l} l = 0, 1, \cdots \\ k = 0, 1, \cdots, n - 1. \end{array} \tag{2.7}$$

Applying the operator (2.5) to (2.3), substituting in (2.1) and comparing coefficients of the powers of x, one finds

$$a_{n+l} = \lambda c_l a_l , \qquad l \geq n \tag{2.8}$$

and hence, for $0 \leq k \leq n - 1$,

$$a_{k+\nu n} = \lambda^\nu a_k \prod_{\mu=0}^{\nu-1} c_{k+n\mu} , \qquad \nu \geq 1. \tag{2.9}$$

Finally, the powers of x for $0 \leq l \leq n - 1$ give

$$\begin{aligned} a_l &= \lambda \sum_{\mu=0}^{n-1} \sum_{\nu=0}^{\infty} A^*_{l,\mu+\nu n} a_{\mu+\nu n} \\ &= \lambda \sum_{\mu=0}^{n-1} \sum_{\nu=0}^{\infty} A^*_{l,\mu+\nu n} \lambda^\nu a_\mu \prod_{\kappa=0}^{\nu-1} c_{\mu+n\kappa} , \qquad l = 0, 1, \cdots , n - 1. \end{aligned} \tag{2.10}$$

In order that the system of equations (2.10) will have non-trivial solutions a_l , one has

$$| \delta_{l\mu} - \lambda G_{l\mu}(\lambda) | = 0, \qquad l, \mu = 0, 1, \cdots , n - 1, \tag{2.11}$$

where

$$G_{l,\mu}(\lambda) = \sum_{\nu=0}^{\infty} A_{l,\mu+\nu n} \lambda^\nu \prod_{\kappa=0}^{\nu-1} c_{\mu+n\kappa} . \tag{2.12}$$

The determinantal equation (2.11) determines the eigenvalues of the integral equation (2.1), while the corresponding eigensolutions are given by

$$\varphi_\lambda(x) = \sum_{\mu=0}^{n-1} \sum_{\nu=0}^{\infty} \lambda^\nu x^{\mu+n\nu} a_\mu \prod_{\kappa=0}^{\nu-1} c_{\mu+n\kappa} . \tag{2.13}$$

It is easily verified from (2.6) that (2.13) converges with ν for all x, the rate of convergence depending on μ and λ. The same is true for the function $G_{l,\mu}(\lambda)$ of (2.12), i.e., for (2.10).

3. Example. As an example of the application of the method of solution of Sec. 2 the following integral equation will be investigated. It is one of the simplest of the type under consideration and in many text books, e.g. Bückner [2], is chosen for the demonstration of methods of solution:

$$\varphi(x) = \lambda\left[\int_0^x (x - y)\varphi(y)\, dy + \int_x^1 (y - x)\varphi(y)\, dy\right]; \tag{3.1}$$

hence, in the notation of Eq. (2.2),

$$P_{n-1}(x, y) = x - y, \qquad i + k = 1, \qquad n = 2, \tag{3.2}$$

$$A_{00} = A_{11} = 0, \qquad A_{01} = -1, \qquad A_{10} = 1.$$

The integral operator in the present case is

$$L(\varphi) = \int_0^x (x - y)\varphi(y)\, dy + \int_x^1 (y - x)\varphi(y)\, dy \tag{3.3}$$

and, in correspondence with (2.5), one has

$$L(y^l) = \frac{2x^{l+2}}{(l+1)(l+2)} - \frac{x}{l+1} + \frac{1}{l+2}, \tag{3.4}$$

since by (2.6)

$$A_{0l}^* = \frac{1}{l+2}, \qquad A_{1l}^* = -\frac{1}{l+1},$$

$$c_l = \frac{2}{(l+1)(l+2)}. \tag{3.5}$$

Proceeding as indicated in Sec. 2, comparison of coefficients of powers of x gives

$$a_{l+2} = \frac{2\lambda a_l}{(l+1)(l+2)}, \qquad l \geq 2, \tag{3.6}$$

$$a_{0+2\nu} = \frac{(2\lambda)^\nu a_0}{(2\nu)!}, \qquad a_{1+2\nu} = \frac{(2\lambda)^\nu a_1}{(2\nu+1)!}, \qquad \nu \geq 1, \tag{3.7}$$

$$\begin{aligned} a_0 &= \lambda\left[a_0 \sum_{\nu=0}^{\infty} \frac{(2\lambda)^\nu}{(2\nu+2)(2\nu)!} + a_1 \sum_{\nu=0}^{\infty} \frac{(2\lambda)^\nu}{(2\nu+3)(2\nu+1)!}\right] \\ a_1 &= -\lambda\left[a_0 \sum_{\nu=0}^{\infty} \frac{(2\lambda)^\nu}{(2\nu+1)!} + a_1 \sum_{\nu=0}^{\infty} \frac{(2\lambda)^\nu}{(2\nu+2)!}\right]. \end{aligned} \tag{3.8a}$$

The system of Eqs. (3.8a) which corresponds to (2.10) is easily seen to lead to the determinantal equation

$$\begin{vmatrix} 1 + \cosh\omega - \omega\sinh\omega & \dfrac{\sinh\omega}{\omega} - \cosh\omega \\ \omega\sinh\omega & 1 + \cosh\omega \end{vmatrix} = 0 \tag{3.9a}$$

or, finally, to

$$2 + 2\cosh\omega - \omega\sinh\omega = 0, \tag{3.9b}$$

where

$$2\lambda = \omega^2. \tag{3.10}$$

The eigensolutions are given by

$$\begin{aligned} \varphi &= a_0 \sum_{\nu=0}^{\infty} \frac{(2\lambda)^\nu x^{2\nu}}{(2\nu)!} + a_1 \sum_{\nu=0}^{\infty} \frac{(2\lambda)^\nu x^{1+2\nu}}{(2\nu+1)!} \\ &= a_0 \cosh\omega x + \frac{a_1}{\omega}\sinh\omega x. \end{aligned} \tag{3.11}$$

4. Conclusions. A method has been developed by which the eigenvalues and eigenfunctions of certain types of Fredholm integral equations of the second kind may be

determined. The method which essentially corresponds to the method of power series, used for ordinary differential equations, may be applied to more general kernels than those of the homogeneous polynomial type considered here. It may also be extended to other systems of linearly independent functions, such as trigonometric functions, depending on the construction of the kernel which in that case would have to involve a finite number of trigonometric terms.

The method should prove of great value in the treatment of vibration problems of systems with polynomial mass distributions and concentrated masses. Two of the simplest problems of this type, whose exact solutions have been known for a large number of years, have been solved by a closely related method in [1].

5. Acknowledgment. The author is indebted to Prof. A. Pfluger at the Eidgenössische Technische Hochschule, Zürich, Switzerland, for some valuable suggestions which greatly assisted the further development of the method.

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ASYMMETRICAL FINITE DIFFERENCE NETWORK FOR TENSOR CONDUCTIVITIES*

By L. TASNY-TSCHIASSNY (*University of Sydney, Australia*)

In a paper recently published in this journal [1] the problem dealt with was described by its author in the following way: "Given a region in which Eq. (4) holds and a large number of points in the region chosen at random, in what way should the points be interconnected with 'physically realizable' electrical resistors in order that the voltages at the nodes shall be as nearly as possible the correct solutions of the boundary value problem characterized by Eq. (4) and appropriate boundary conditions?"

The Eq. (4) quoted is the special case of the differential equation

$$\nabla \cdot \{\sigma[\epsilon] \cdot \nabla \Phi\} + \tau = 0 \tag{1}$$

for $[\epsilon]$ being the unit tensor, i.e., for isotropically conducting material. In Eq. (1) Φ is the electrical potential in a two-dimensional continuum, τ the current density of a distributed transverse external source, σ the conductivity of the continuum in a certain direction, and $[\epsilon]$ the non-uniformity tensor of the conductivity. The scalars τ, σ, and the tensor $[\epsilon]$ may be functions of the position.

The author of the present Note had been interested in a practically identical problem for which he had coined the expression "triangulation of a two-dimensional continuum" and which included the general case of tensor conductivities [2]. He replaced the interior

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of a triangle of any suitable shape and small size by a delta of resistors (see Fig. 1) so that, with the exception of the boundary, two resistors in parallel always connected a node pair in the final equivalent network. If the principal (maximum or minimum)

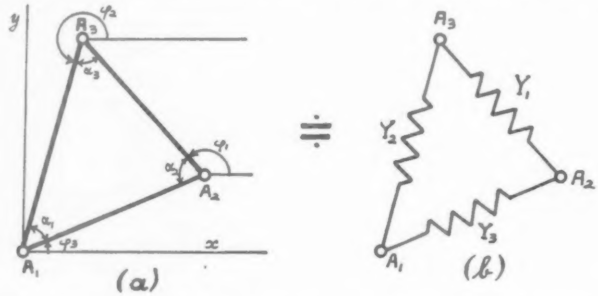

FIG. 1. Replacement of the interior of a triangle by a delta of resistors

conductivities are denoted by $\sigma = \sigma_x$ in the x-direction and $\sigma_y = k\sigma_x$ in the y-direction, so that

$$[\epsilon] = \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix},$$

the formula

$$2Y_3/\sigma_x = \frac{1+k}{2} \cdot \cot \alpha_3 + \frac{1-k}{2} \cdot \frac{\cos(\varphi_1 + \varphi_2)}{\sin \alpha_3} \tag{2}$$

(and similarly for Y_1 and Y_2) resulted, which for $k = 1$, i.e., isotropic materials, is identical with MacNeal's Eq. (10). With a proper lay-out of the nodes it can, for tensor conductivities also, be easily arranged that sides of triangles coincide with the boundaries and that all resulting conductances connecting two nodes are positive, i.e., physically realizable.

MacNeal's paper embodies two remarkable advances in the solution of the problem in question. One is finding that an infinite multitude of simulating networks exists for given node points, and the other is the precise definition of the part of the continuum to be lumped at a node as far as the distributed external current sources are concerned. These advances can be extended to tensor conductivities, quite frequently met with in engineering, along the line of MacNeal's method.

In Fig. 2, which corresponds to a part of Fig. 4 in MacNeal's paper, let A and B be two nodes of the simulating network and 1 and 2 the corresponding points of the dual network so that the current I_{AB} crossing the line 12 within the two-dimensional continuum divided by the potential difference V_{AB} between the points A and B determines the conductance Y_{AB} of the resistor joining A and B. Let the direction 12 be found from the direction AB in the following way: Draw the line 34 perpendicular to AB. Multiply the distance $N4$ from the y-axis of a suitable point 4 of this line by $1/k$. Make Nx' parallel to $0x$ and $N2$ equal to $1/k \cdot N4$. Using MacNeal's notation we obtain for I_{AB}

$$I_{AB} = -\int_1^2 \sigma\, dr\, \mathbf{n} \cdot \{[\epsilon]\nabla\phi\} = -\int_1^2 \sigma\, dr \{\mathbf{n}[\epsilon]\} \cdot \nabla\phi \cong -\sigma \cdot r_{34} \mid \nabla\phi \mid_0 \cos\alpha \tag{3}$$

so that

$$Y_{AB} = \sigma \frac{r_{34}}{l_{AB}} = \sigma \frac{r_{30}}{l_{AB}} + \sigma \frac{r_{04}}{l_{AB}}. \tag{4}$$

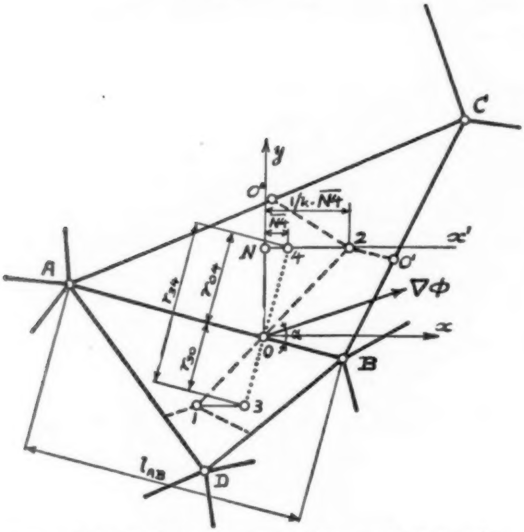

FIG. 2. Portion of the asymmetrical network of triangles.

This relation is general, whether 0 is the mid-point of AB or not. Equation 4 shows the splitting-up of Y_{AB} into two partial conductances corresponding to the interior of the two triangles ABC and ABD.

A closer investigation for 0 as the mid-point of AB is carried out in the notations of Fig. 3. The bisectors H_1L, H_2L and H_3L of the sides of the triangle $A_1A_2A_3$, with

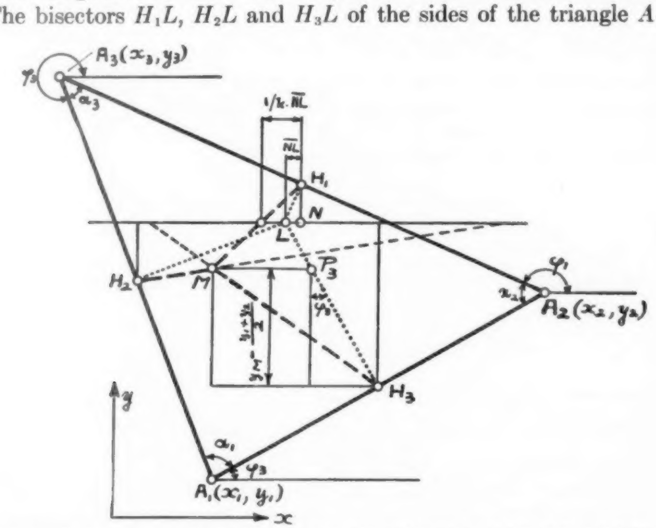

FIG. 3. Conditions for 0, 0′, and 0″ of Fig. 2 being the mid-points of AB, BC and CA respectively.

the Cartesian co-ordinates of A_1, A_2 and A_3 being (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively, intersect at L. Let the lines H_3M and H_1M be the lines derived from the lines H_3L and H_1L by the procedure by which in Fig. 2 the line 12 was derived from the line 34. Let them intersect at the point $M(x_M, y_M)$. Using basic methods of analytical geometry we obtain for y_M

$$2y_M = \frac{k(x_3 - x_2)(x_2 - x_1)(x_1 - x_3) - [y_3^2(x_2 - x_1) + y_2^2(x_1 - x_3) + y_1^2(x_3 - x_2)]}{x_3(y_2 - y_1) + x_2(y_1 - y_3) + x_1(y_3 - y_2)} \tag{5}$$

and a similar formula for x_M. These formulae are symmetrical in the co-ordinates of A_1, A_2 and A_3, so that the line H_2M derived from the line H_2L by the same procedure passes through M. In other words, if the points 0, 0′ and 0″ in Fig. 2 are the mid-points of the sides of the triangle ABC, the line 02 for the side AB and the corresponding lines 0′2 and 0″2 for the sides BC and CA intersect in one point 2.

If (see Fig. 3) the ratio

$$Y_3 = \sigma \frac{H_3P_3}{A_1A_2} = \sigma \cdot \frac{y_M - (y_1 + y_2)/2}{x_2 - x_1} \tag{6}$$

that corresponds to the ratio $\sigma \cdot r_{04}/l_{AB}$ of Fig. 2 is computed, we obtain after some manipulating

$$2Y_3 = \frac{\sigma}{\cot \varphi_2 - \cot \varphi_1} + \frac{k\sigma}{\tan \varphi_1 - \tan \varphi_2}. \tag{7}$$

Equation (7) is identical with Eq. (2), as can easily be shown. Hence Eq. (2) represents the special case of Eq. (4), if the point 0 (Fig. 2) and the corresponding other two points 0′ and 0″ are the mid-points of the sides of the triangle ABC.

Tensor conductivities can be dealt with also by a change of variables [3]. This method leads to the same results.

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IMPEDANCE SYNTHESIS WITHOUT MUTUAL COUPLING*

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In a fundamental contribution to network theory, Bott and Duffin [1] have given a method for the synthesis of an impedance which obviates the use of any mutual coupling

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as was formerly required by the Brune realization technique. The purpose of this note is to present a modification of their procedure which results in fewer elements.

The Bott-Duffin method depends upon the result [3] that if $Z(p)$ is a positive real rational function (p.r.f.) of degree n, then

$$R(p) = \frac{pZ(p) - kZ(k)}{pZ(k) - kZ(p)}, \qquad k > 0 \tag{1}$$

is again a p.r.f. of degree not greater than n. Using (1), they show that $Z(p)$ is realized by a balanced bridge whose opposite pairs of arms are $Z(k)R(p)$, $Z(k)/R(p)$ and $Z(k)p/k$, $kZ(k)/p$ respectively. If now it is assumed (after well-known preliminary reductions) that at $p = i\omega_0$, $\omega_0 > 0$, $Z(i\omega_0) = i\omega_0 L$, $L > 0$,[1] then by choosing $k > 0$ such that $Z(k)/k = L$ they obtain an $R(p)$ which has a pole at $p = \pm i\omega_0$, so that

$$R(p) = R_1(p) + \frac{\alpha p}{p^2 + \omega_0^2}, \qquad \alpha > 0, \tag{2}$$

where $R_1(p)$ is a p.r.f. two degrees lower than $R(p)$. Thus their algorithm requires six elements and two reduced functions.

Our alternative synthesis of the above case utilizes an unbalanced bridge network Γ whose pairs of opposite arms are Z_1, Z_4 and Z_2, Z_3 respectively, and whose bridging arm Z_5 connects the common node of Z_1 and Z_2 and the common node of Z_3 and Z_4. We shall assume that $Z_1 = W + S$, $Z_4 = 1/W$ where W and S are both p.r.f. If $Z'(p)$ is the impedance of Γ, then (see [2, pp. 284-285])

$$\begin{aligned} Z'(p) = \{&Z_3(Z_2 + Z_5)W^2 + [Z_3S(Z_2 + Z_5) + Z_5(Z_2Z_3 + 1) + Z_2 + Z_3 + Z_5]W \\ &+ S(Z_2 + Z_3 + Z_5) + Z_2(Z_3 + Z_5)\}/\{(Z_2 + Z_5)W^2 + [S(Z_2 + Z_5) \\ &+ Z_2Z_3 + Z_2Z_5 + Z_3Z_5 + 1]W + S + Z_3 + Z_5\} \end{aligned} \tag{3}$$

We will now show how to choose Z_2, Z_3, Z_5, S and a reduced function W so that $Z'(p) = Z(p)$. It is desirable to impose the further condition that the right member of (3) reduce to a bilinear function of W. For this to be so, the resultant[2] of the two quadratic polynomials which are the numerator and denominator of the fraction in (3) must be zero. After some calculation this leads to the equation

$$S = \frac{(1 - Z_2Z_3)(Z_3 + Z_5)}{Z_3(Z_2 + Z_5)}. \tag{4}$$

The simplest reactive choices of the Z's in (4) which cause the resistors in W and $1/W$ to be eliminated at the frequency $p = \pm i\omega_0$ and which are consistent with $Z(i\omega_0)/i > 0$ are

$$Z_2 = \frac{a\omega_0^2}{p}, \qquad Z_3 = bp, \qquad Z_5 = ap, \qquad a > 0,\ b > 0;$$

which gives

$$S = \frac{(1 - ab\omega_0^2)(a + b)p}{ab(p^2 + \omega_0^2)},$$

[1] If $L < 0$, one considers either $1/Z(p)$ or $Z(\omega_0^2/p)$ instead of $Z(p)$.

[2] The resultant of $a_0W^2 + a_1W + a_2$ and $a_0'W^2 + a_1'W + a_2'$ is $(a_1'a_2 - a_1a_2')(a_0'a_1 - a_0a_1') - (a_0a_2' - a_0'a_2)^2$.

where a and b must be chosen so that $1 - ab\omega_0^2 > 0$. Substituting these values of the Z's and S in (3), dividing out the common factor

$$W + \frac{Z_3 + Z_5}{Z_3(Z_2 + Z_5)}$$

and simplifying, we obtain

$$Z'(p) = \frac{abp(p^2 + \omega_0^2)W + (a + b - ab^2\omega_0^2)p^2 + a\omega_0^2}{a(p^2 + \omega_0^2)W + p(abp^2 + 1)}. \tag{5}$$

Now it follows from (1) and (2) that $Z(p)$ may be written in terms of the reduced function $R_1(p)$ as

$$Z(p) = Z(k) \cdot \frac{p(p^2 + \omega_0^2)R_1 + (\alpha + k)p^2 + k\omega_0^2}{k(p^2 + \omega_0^2)R_1 + p(p^2 + \alpha k + \omega_0^2)}. \tag{6}$$

A comparison of (5) and (6) shows that if we let $W = R_1$, $b = 1/k$, $a = k/(\alpha k + \omega_0^2)$, we obtain $Z'(p) = Z(p)/Z(k)$. Making this impedance level change in $Z'(p)$, we conclude that $Z(p)$ may be realized as an unbalanced bridge with

$$Z_1 = W + S, \qquad W = Z(k)R_1(p), \qquad S = \frac{Z(k)(1 - ab\omega_0^2)(a + b)p}{ab(p^2 + \omega_0^2)};$$

$$Z_2 = a\omega_0^2 Z(k)/p, \qquad Z_3 = bZ(k)p, \qquad Z_4 = Z(k)/R_1(p), \qquad Z_5 = aZ(k)p,$$

$$a = k/(\alpha k + \omega_0^2), \qquad b = 1/k.$$

Thus our algorithm requires five elements and two reduced functions.

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ON TWO METHODS OF GENERATING SOLUTIONS OF LINEAR PARTIAL DIFFERENTIAL EQUATIONS BY MEANS OF DEFINITE INTEGRALS†

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1. Definite integrals with variable limits of integration. Consider the canonical form of the linear hyperbolic differential equation in two independent variables

$$L(u) \equiv u_{xy} + a(x, y)u_x + b(x, y)u_y + c(x, y)u = 0. \tag{1}$$

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It has been well known since the beginnings of the subject (see the detailed discussion in Goursat [4]) that if $U(x, y, \alpha)$ is any[1] solution of (1) containing a parameter α, for $\alpha_1 \leq \alpha \leq \alpha_2$ with α_1 and α_2 fixed constants, then

$$u(x, y) = \int_{\alpha_1}^{\alpha_2} f(\alpha) U(x, y, \alpha) \, d\alpha, \tag{2}$$

where $f(\alpha)$ is an arbitrary function, is also a solution of (1). In this way solutions of (1) can be generated from functions $f(\alpha)$ of a single variable α.

In 1895, Le Roux [1] showed that one or both of the fixed limits of integration can be replaced by a characteristic variable [say, x or y, in the case of (1)], and that (2) will still be a solution of (1), provided only that $U(x, y, \alpha)$ is chosen so as to satisfy certain additional conditions. Thus, if either limit of integration is replaced by x, then $U(x, y, \alpha)$ must satisfy the characteristic condition

$$\frac{\partial U}{\partial y} + a(\alpha, y) U = 0 \qquad \text{on } x, = \alpha, \tag{3}$$

which, incidentally, is one of the conditions satisfied by the Riemann function. Similarly, if either limit of integration is replaced by y, then $U(x, y, \alpha)$ must satisfy

$$\frac{\partial U}{\partial x} + b(x, \alpha) U = 0, \qquad \text{on } y = \alpha. \tag{4}$$

As a matter of fact, the characteristic variables x and y, or an arbitrary function of just one of them, are the only variable limits which may be used in the definite integral in (2), if $u(x, y)$ is to be again a solution of (1). This is proved by Le Roux [1] for the case in which the functions occurring in (2) are such that the definite integral may be differentiated under the integral sign.

For instance, if $R(\xi, \eta; x, y)$ is the Riemann function[2] of (1) [so that, for fixed (ξ, η), the function $R(\xi, \eta; x, y)$ of the "pole variables" (x, y) is a solution of (1)] then, for η fixed, $R(\alpha, \eta; x, y)$ will be a solution of (1) satisfying (3); while, for ξ fixed, $R(\xi, \alpha; x, y)$ will be a solution of (1) satisfying (4); and, finally, $R(\alpha, \alpha; x, y)$ will be a solution of (1) satisfying both (3) and (4). Accordingly, for arbitrary f the definite integrals with variable limits

$$\int_{\alpha_1}^{x} f(\alpha) R(\alpha, \eta; x, y) \, d\alpha, \qquad \int_{\alpha_1}^{y} f(\alpha) R(\xi, \alpha; x, y) \, d\alpha, \qquad \int_{x}^{y} f(\alpha) R(\alpha, \alpha; x, y) \, d\alpha,$$

will each be a solution of (1).

2. The integral operator. In several papers Bergman[3] has developed a theory of integral operators for obtaining solutions of partial differential equations. As applied to (1), the method is, for the most part (see the concluding paragraph of Sec. 3), based on the following theorem:

[1] Throughout this paper precise statements concerning differentiability, continuity, and regions of definition of functions will be omitted, except in those cases where they are of prime importance.

[2] For the connection between the integral operator of the next section and Riemann's function, see [10].

[3] For a summary and bibliography see in particular [9] and [11].

Theorem. Let $E(x, y, t)$ be a solution of

$$(1 - t^2)(E_{yt} + aE_t) - \frac{1}{t}(E_y + aE) + 2xtL(E) = 0, \tag{5}$$

such that, for $x \neq 0$,

$$\frac{(1 - t^2)^{1/2}(E_y + aE)}{xt} \tag{6}$$

is continuous for $t = 0$, and tends to zero for each (x, y) as t approaches ± 1. Then, if f is an arbitrary once continuously differentiable function, the function u defined by

$$u(x, y) = \int_{-1}^{+1} E(x, y, t) f[\tfrac{1}{2}x(1 - t^2)] \frac{dt}{(1 - t^2)^{1/2}} \tag{7}$$

is a solution of (1).

Proof. Writing, for brevity, f instead of $f[\frac{1}{2}x(1 - t^2)]$, one has, upon differentiating (7),

$$L(u) = \int_{-1}^{+1} \left\{ L(E) \cdot f + (E_y + aE) \cdot \frac{\partial f}{\partial x} \right\} \frac{dt}{(1 - t^2)^{1/2}} \cdot \tag{8}$$

Now,

$$\frac{1}{2}(1 - t^2)\frac{\partial f}{\partial t} + xt\frac{\partial f}{\partial x} = 0,$$

and when $xt \neq 0$ this equation may be solved for $\partial f/\partial x$. Substituting for $\partial f/\partial x$ in (8) and integrating by parts (*cf.* (6) at $t = 0$), one obtains

$$L(u) = \int_{-1}^{+1} \left\{ \frac{L(E)}{(1 - t^2)^{1/2}} + \left[\frac{(1 - t^2)^{1/2}}{2xt}(E_y + aE) \right]_t \right\} f\, dt - \left[\frac{(1 - t^2)^{1/2}}{2xt}(E_y + aE) f \right]_{-1}^{+1} .$$

The desired conclusion now follows from the hypothesis made about (6) as $t \to \pm 1$, and the fact that

$$\frac{L(E)}{(1 - t^2)^{1/2}} + \left[\frac{(1 - t^2)^{1/2}}{2xt}(E_y + aE) \right]_t = 0,$$

by virtue of (5).

This theorem, as stated and proved above for Eq. (1), does not seem to be stated explicitly in full in the literature. However, for an equation occurring in gas dynamics [when $a = b = 0$, $c = c(x - y)$], see Bergman [6]. Moreover, Ghaffari [8] considers the general equation (1), but does not state the essential conditions on the behavior of (6) at $t = 0$.

3. Definite integrals with variable limits of integration and the integral operator. The object of the present note is to show that the theorem of Sec. 2 and its obvious modification when x and y are complex (which constitute the kernel of Bergman's operator method), are merely a restatement of part of Le Roux's results mentioned in Sec. 1, and that Le Roux's formulation is simpler in the fluid dynamical application.

First of all, notice that $E(x, y, t)$ can be considered—without any loss of generality—to be an even function of t, for otherwise $E(x, y, t)$ may be replaced by its even part $\frac{1}{2}[E(x, y, t) + E(x, y, -t)]$ in (7), without altering the value of the definite integral. Hence, instead of (7), consider

$$u(x, y) = 2 \int_0^1 E(x, y, t) f[\tfrac{1}{2}x(1 - t^2)] \frac{dt}{(1 - t^2)^{1/2}} \cdot \tag{9}$$

Then, for $x \neq 0$, the change of integration variable

$$\alpha = x(1 - t^2), \qquad \text{i.e.} \qquad t = \left(\frac{x - \alpha}{x}\right)^{1/2}, \tag{10}$$

transforms Eq. (9) into

$$u(x, y) = \int_0^x E\left[x, y, \left(\frac{x - \alpha}{x}\right)^{1/2}\right] f(\tfrac{1}{2}\alpha) \frac{d\alpha}{[\alpha(x - \alpha)]^{1/2}} \cdot$$

Now, if $U(x, y, \alpha)$ is defined as

$$U(x, y, \alpha) = \frac{E\left[x, y, \left(\frac{x - \alpha}{x}\right)^{1/2}\right]}{[\alpha(x - \alpha)]^{1/2}},$$

so that

$$\frac{\partial U}{\partial x} = \frac{1}{[\alpha(x - \alpha)]^{1/2}} \left\{E_x + \frac{\alpha}{2x^{3/2}(x - \alpha)^{1/2}} E_t - \frac{1}{2(x - \alpha)} E\right\},$$

$$\frac{\partial U}{\partial y} = \frac{1}{[\alpha(x - \alpha)]^{1/2}} E_y ,$$

$$\frac{\partial^2 U}{\partial x\, \partial y} = \frac{1}{[\alpha(x - \alpha)]^{1/2}} \left\{E_{xy} + \frac{\alpha}{2x^{3/2}(x - \alpha)^{1/2}} E_{yt} - \frac{1}{2(x - \alpha)} E_y\right\},$$

then

$$\begin{aligned} L(U) &= \frac{1}{[\alpha(x - \alpha)]^{1/2}} \left\{L(E) + \frac{\alpha}{2x^{3/2}(x - \alpha)^{1/2}} (E_{yt} + aE_t) - \frac{1}{2(x - \alpha)} (E_y + aE)\right\}, \\ &= \frac{1}{[\alpha(x - \alpha)]^{1/2}} \left\{L(E) + \frac{\alpha/x}{2x[(x - \alpha)/x]^{1/2}} (E_{yt} + aE_t) \right. \\ &\qquad \left. - \frac{1}{2x[(x - \alpha)/x]} (E_y + aE)\right\}, \\ &= 0, \end{aligned}$$

by (5), since the t argument of E is just $[(x - \alpha)/x]^{1/2}$. Hence $U(x, y, \alpha)$ is a solution of (1) containing a parameter α. Further

$$\left(\frac{\partial U}{\partial y} + aU\right) = \frac{1}{[\alpha(x - \alpha)]^{1/2}} (E_y + aE),$$

so that

$$\frac{(1 - t^2)^{1/2}}{xt} (E_y + aE) = \frac{\alpha}{x}\left(\frac{\partial U}{\partial y} + aU\right).$$

Now, the theorem uses the continuity, at $t = 0$, of the function on the left hand side of the last equation. At first glance this appears to be a weaker requirement than the vanishing of the right hand side at $\alpha = x$, which is needed by Le Roux. However, as has already been remarked, E can be assumed to be an even function of t, so that $(1 - t^2)^{1/2}(E_y + aE)/xt$ is odd in t; hence the continuity of the latter function at $t = 0$ implies that it must of necessity vanish there.

Thus the function $E(x, y, t)$ used by Bergman may be written as the sum of an even part $[E(x, y, t) + E(x, y, -t)]/2$, which satisfies Le Roux's condition (3) after the transformation (10) is performed; and an odd part $[E(x, y, t) - E(x, y, -t)]/2$, which plays no essential role, since it contributes nothing to the definite integral. This remark about the odd part of E has apparently been overlooked in working out some particular examples.

If the independent variables x and y in (1) are both *complex*, and the coefficients a, b, c are analytic functions of these two complex variables, then there is no essential distinction between equations of hyperbolic and elliptic type. In this case, which is the one usually considered by Bergman, the function E is taken to be analytic in the two complex variables x and y, and the function f (since it is differentiated with respect to x) is taken to be an analytic function of its (complex) argument. The formal calculation leading to the proof of the theorem is exactly the same as in the real case considered above. By this passage to complex x and y, however, the class of non-analytic solutions of a hyperbolic differential equation is excluded from direct consideration. It should also be remarked, see [5], that for certain purposes it is convenient, in the case when x and y are both complex, to consider t as complex also, and to integrate from $t = -1$ to $t = +1$ along an arbitrary curve in the t-plane. If this curve does not pass through the origin, the Le Roux condition (3) need not be fulfilled, and hence, in these circumstances, Bergman's method is not equivalent to Le Roux's. However, in concrete applications the path of integration from $t = -1$ to $t = +1$ seems to be invariably taken along the real axis, as in the theorem of Sec. 2.

4. The application of the integral operator method to fluid dynamics. One of the major applications of Bergman's integral operator method is to the equations (in a "modified hodograph" plane) governing the plane steady irrotational flow of a perfect compressible fluid. Under a suitable change of independent variables the equation for, say, the stream function ψ, is of the type (1), with coefficients a, b, c depending on the difference $x - y$ alone.

The method of Le Roux reduces in this case to finding a solution $U^*(x, y)$ of (1) which satisfies either the condition (3) on the *single* characteristic $x = 0$, or the condition (4) on the *single* characteristic $y = 0$, or both. Once this solution $U^*(x, y)$ has been found, one sets

$$U(x, y, \alpha) = U^*(x - \alpha, y - \alpha), \tag{11}$$

and then a family of solutions depending on one arbitrary function is given at once by (2), with α_1 and/or α_2 suitably replaced by x and/or y, as the case may be. For, since the partial differential equation is invariant when the variables x and y are replaced by $x - \alpha$ and $y - \alpha$ respectively, the function $U(x, y, \alpha)$ defined by (11) is again a solution, and will satisfy the corresponding Le Roux conditions on $x = \alpha$ and/or $y = \alpha$.

A special case of Eq. (1) with coefficients depending on $x - y$ alone is the Euler-Poisson equation (see Darboux [2])

$$u_{xy} - \frac{\beta'}{x - y} u_x + \frac{\beta}{x - y} u_y = 0, \tag{12}$$

which for $\beta = \beta' = 1/6$ serves as a first approximation (Tricomi) to the equation for the stream function of compressible flow near the sonic line. Darboux uses the procedure just outlined in the previous paragraph, and an argument equivalent to that of Le Roux, to construct the general solution of Eq. (12); and from this, Tricomi [3], has constructed a simple definite integral representation for the solution of the Cauchy problem for (12) with $\beta = \beta' = 1/6$, the function u and its normal derivative being given on the sonic line $x = y$.

Besides the general compressibility equation, Bergman considers (in his terminology) a "simplified" compressibility equation [7]. In effect this "simplified" equation is that of Tricomi, under change of variables. In dealing with this "simplified" equation, he first finds solutions of the corresponding equation (5) for E in terms of hypergeometric functions, and is led to a complicated representation of a class of solutions of this equation.

Thus, even for this "simplified" equation, it appears that Bergman's method is more complicated than that of Le Roux. Indeed, Bergman's method consists in finding particular solutions of an associated equation for E, Eq. (5), in three independent variables, whereas the method of Le Roux consists in determining particular solutions of the originally given differential equation in two independent variables.

The complications involved are apparent from the following concluding remark: Bergman looks for particular functions $E(x, y, t)$ which involve series whose general term is a power of t^2x times a function of $x - y$. This is easily seen to be equivalent, in view of the change of variable (10), to just looking for particular solutions of the original differential equation using series whose general term is a power of $(x - \alpha)$ times a function of $x - y$, or (because of the invariance of the differential equation when x and y are replaced by $x - \alpha$ and $y - \alpha$ respectively), just a power of x times a function of $x - y$.

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AN INTEGRAL EQUATION GOVERNING ELECTROMAGNETIC WAVES*

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1. Preliminaries. We shall consider in this paper the Helmholtz equation

$$\Delta u + k^2 u = 0 \tag{1}$$

in two independent variables ξ and η. In electromagnetic theory the problem arises of determining, in the exterior D of a simple closed curve C, a solution u of (1) with prescribed boundary values or prescribed normal derivatives on C. The function u is supposed in addition to fulfil at infinity the radiation condition

$$\lim_{r \to \infty} r^{1/2}\left(\frac{\partial u}{\partial r} - iku\right) = 0, \tag{2}$$

where $r = (\xi^2 + \eta^2)^{1/2}$ is the distance from the origin. Thus u will be complex-valued and $u(\xi, \eta)e^{-ikt}$ will be the actual wave-function for the problem considered. We can think of u as representing a two-dimensional wave scattered by the finite object C, and the type of boundary condition to be imposed along C then depends on whether we treat the electric or the magnetic field.

We shall investigate in detail the case when the boundary values of u are given along C. However, all our statements carry over almost verbatim when it is the normal derivatives which are prescribed instead. The uniqueness of the solution of (1) and (2) follows easily from the work of Rellich [2], but the existence of the solution as presented by Weyl [3] and Müller [1] depends on a careful discussion of the eigenfunctions of (1) for the interior of C. The object of the present note is to reduce the boundary value problem for (1) in the exterior domain D to a new Fredholm integral equation whose study is independent of the interior domain and its eigenvalues. This will be accomplished by introducing a suitable parametrix for the problem, constructed by means of conformal mapping. Our integral equation has, furthermore, the advantage that it can be solved numerically by iteration in certain important cases.

The middle section of the paper is devoted to the development of this integral equation. In the concluding section, we discuss the uniqueness question from a new point of view and indicate a variational formula for the estimation of the scattering cross section of C.

We shall assume that the frequency $k = 1$ in the following without loss of generality, since this reduction corresponds merely to a change of scale.

2. The parametrix method. Our analysis of the equation (1) is based on the notion of a parametrix $S(\zeta, \tau)$. Such a parametrix S is not a solution of (1), but it satisfies the following requirements. It is regular as a function of ζ in D, except at $\zeta = \tau$, where

$$S(\zeta, \tau) + \log | \zeta - \tau |$$

remains continuous together with its first partial derivatives. At infinity S satisfies the radiation condition

$$\lim_{r \to \infty} r^{1/2}\left(\frac{\partial S}{\partial r} - iS\right) = 0$$

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as a function of ζ, and along C we have $S = 0$ for each fixed τ in D.

Suppose that u is any solution of (1) and (2) in D with $k = 1$, as was agreed above. Then from the radiation condition we obtain

$$\lim_{r\to\infty} \int_{|\zeta|=r} \left(u \frac{\partial S}{\partial n} - S \frac{\partial u}{\partial n} \right) ds = 0,$$

where s and n represent arc length and inner normal along the path of integration. Hence by Green's theorem and by (1),

$$\int_C u(\zeta) \frac{\partial S(\zeta, \tau)}{\partial n} ds = 2\pi u(\tau) - \iint_D u(\Delta S + S) \, d\xi \, d\eta. \tag{3}$$

There are many elementary ways in which S can be constructed, and in each case (3) provides an integral equation for the determination of u.

We choose S in a special and particularly useful manner. Let

$$\zeta = z + f(z) \tag{4}$$

be the conformal transformation of the exterior $|z| > R$ of a circle of radius R onto D, normalized so that

$$f(z) = \sum_{m=0}^{\infty} \frac{a_m}{z^m} \tag{5}$$

is regular at infinity. Furthermore, denote by $G(z, w)$ the Green's function for (1) in the region $|z| > R$. To be precise, G satisfies the equation

$$\Delta G + G = 0 \tag{6}$$

as a function of z, except at $z = w$, where

$$G(z, w) + \log |z - w|$$

remains continuous; G satisfies the radiation condition at ∞; and $G = 0$ for $|z| = R$. This particular Green's function has a variety of explicit representations in terms of Bessel functions, all of which are obtained by the standard procedure of separation of variables in polar coordinates.

With ζ given by (4) and with $\tau = w + f(w)$, we set

$$S(\zeta, \tau) = G(z, w). \tag{7}$$

The function S so constructed by application of a conformal transformation on the arguments of the known Green's function G is evidently a parametrix of the type defined above. Indeed, S is regular in D except for a logarithmic infinity at $\zeta = \tau$, S vanishes on C, and S even satisfies the radiation condition, since the conformal transformation (4) has a derivative 1 at infinity and therefore leaves this condition invariant.

We substitute (7) into (3) and evaluate the integrals in the z-plane. The area element $d\xi\, d\eta$ and the Laplacian ΔS are altered in the transformation by multiplication and division, respectively, with the Jacobian

$$\frac{\partial(\xi, \eta)}{\partial(x, y)} = |1 + f'(z)|^2.$$

In particular, the expression $\Delta S\, d\xi\, d\eta$ is a conformal invariant, as is also $(\partial S/\partial n)\, ds$. Therefore we obtain

$$\int_{|z|=R} U(z) \frac{\partial G(z, w)}{\partial n} ds = 2\pi U(w) - \iint\limits_{|z|>R} U(z)[\Delta G + G \mid 1 + f' \mid^2]\, dx\, dy,$$

where $U(w)$ stands for $u(\tau)$ and likewise

$$U(z) = u(\zeta). \tag{8}$$

Application of (6) yields

$$\int_{|z|=R} U(z) \frac{\partial G(z, w)}{\partial n} ds = 2\pi U(w) - \iint\limits_{|z|>R} U(z) G(z, w)\{2\, \mathcal{R}e\, f'(z) + \mid f'(z) \mid^2\}\, dx\, dy. \tag{9}$$

In order to derive from (9) a Fredholm integral equation for U with a symmetric kernel, we introduce the notations

$$p(z) = \{2\, \mathcal{R}e\, f'(z) + \mid f'(z) \mid^2\}^{1/2}, \tag{10}$$

$$V(z) = p(z) U(z), \tag{11}$$

$$K(z, w) = \frac{1}{2\pi} p(z) p(w) G(z, w). \tag{12}$$

We notice that when $u(\zeta)$ is prescribed on C, the expression

$$g(w) = \frac{p(w)}{2\pi} \int_{|z|=R} U(z) \frac{\partial G(z, w)}{\partial n} ds \tag{13}$$

will be known. Multiplying (9) on both sides by $p(w)/2\pi$, we arrive at the final Fredholm integral equation

$$g(w) = V(w) - \iint\limits_{|z|>R} V(z) K(z, w)\, dx\, dy \tag{14}$$

for V, or in other words for U, and this is the basis of our entire discussion.

With $w = a + ib$, the symmetric kernel K of (14) satisfies the square-integrability condition

$$\iint\limits_{|w|>R} \iint\limits_{|z|>R} \mid K(z, w) \mid^2 dx\, dy\, da\, db$$
$$= \frac{1}{4\pi^2} \iint\limits_{|w|>R} \iint\limits_{|z|>R} \mid G(z, w) \mid^2 \mid p(z) p(w) \mid^2 dx\, dy\, da\, db < \infty, \tag{15}$$

since $\mid z \mid p(z)$ is bounded at infinity, by (5) and (10), and since $\mid z \mid^{1/2} \mid w \mid^{1/2} G(z, w)$ is bounded outside a fixed neighborhood of $z = w$, by standard estimates of the Bessel functions. Hence the Fredholm theory is immediately applicable to (14), and we deduce that for any given g we can find a unique square-integrable solution V, provided that the homogeneous equation

$$0 = V - \iint\limits_{|z|>R} VK\, dx\, dy \tag{16}$$

has no non-trivial eigenfunction V. But an eigenfunction V transforms by (11) and (8) into a solution u of (1) which vanishes on C and satisfies the radiation condition (2), as can easily be verified by direct calculation from (16). Rellich's uniqueness theorem [2] then shows that $u \equiv 0$, whence $V \equiv 0$ and there are, indeed, no eigenfunctions.

Thus, given u on C, we can calculate g, substitute into (14), solve for V, and find u as a solution of (1) and (2) by using (8) and (11). This completes our proof of the existence of the solution of the boundary value problem. The hypotheses on the smoothness of C which are required in the argument are only those needed for the demonstration of the uniqueness theorem. In particular, our integral equation yields the solution of the problem for curves C with a finite number of corners, and the behavior of u at the corners can be derived from (14). Furthermore, the same proof is valid when it is the normal derivatives of u which are assigned on C. For this case, we have only to replace the Green's function G throughout the argument by the corresponding Neumann's function $N(z, w)$ for (1) and (2) in the region $|z| > R$. The function N is characterized by the same requirements as we imposed upon G, except that we replace the boundary condition $G = 0$ by the boundary condition $\partial N/\partial n = 0$ along $|z| = R$. For the circle, the Neumann's function has an explicit expansion in terms of Bessel functions, obtained by separation of variables.

From the practical standpoint, the integral equation (14) has some interest. If the domain D differs sufficiently little from the exterior of a circle, the function p depending on the conformal transformation (4) will be so small that

$$\iint_{|z|>R} |K(z, w)| \, dx \, dy \leq \epsilon < 1. \tag{17}$$

Hence the smallest eigenvalue of (14) will have a modulus exceeding 1 and the resolvent kernel can be used to solve the equation in a Neumann-Liouville series. In fact, the successive approximations defined by

$$V_m = g + \iint_{|z|>R} V_{m-1} K \, dx \, dy \tag{18}$$

converge geometrically when (17) holds. The *a priori* bound

$$|V| \leq \frac{\max |g|}{1 - \epsilon}, \tag{19}$$

which follows immediately in this case, is also significant, since the maximum principle fails for the partial differential equation (1) and more elementary estimates of u are therefore not available.

The case (17) of convergence of the successive approximations (18) to the solution V of (14) has an implication for the existence of solutions of the boundary value problem for (1), since no hypothesis of smoothness on the boundary curve C is required when (17) holds. Finally, we point out that in the development of our method, the conformal transformation (4) and the explicit Green's function G could be replaced by the corresponding quantities for any region in which one knows how to solve the equation (1).

3. Remarks about uniqueness and the scattering cross section. For solutions u of

the Helmholtz equation in D not necessarily satisfying the radiation condition, one can define a norm by the formula

$$|| u ||^2 = \lim_{r\to\infty} \int_{|z|=r} \left\{ | u |^2 + \left| \frac{\partial u}{\partial n} \right|^2 \right\} ds. \tag{20}$$

If u_+ is the solution of (1) and (2) with boundary values h on C and if u_- is the solution of (1) and (2) with boundary values $\bar{h}$ on C, it then follows that the function

$$u_0 = \frac{u_+ + \bar{u}_-}{2} \tag{21}$$

is characterized by the property that among all solutions u of (1) with the boundary values h on C it has the smallest norm,

$$|| u_0 || = \min || u ||. \tag{22}$$

Indeed, writing $v = u - u_0$, we find

$$|| u ||^2 = || u_0 ||^2 + || v ||^2,$$

since by the radiation condition

$$\lim_{r\to\infty} \int_{|z|=r} \left\{ u_0\bar{v} + \frac{\partial u_0}{\partial n} \frac{\partial \bar{v}}{\partial n} \right\} ds = \frac{1}{2i} \int_C \left\{ \bar{v}\left(\frac{\partial u_+}{\partial n} - \frac{\partial \bar{u}_-}{\partial n} \right) - (u_+ - \bar{u}_-) \frac{\partial \bar{v}}{\partial n} \right\} ds$$
$$= 0.$$

The radiation condition also yields for the minimum value of the norm the relation

$$|| u_0 ||^2 = \lim_{r\to\infty} \int_{|z|=r} | u_+ |^2 \, ds$$
$$= \lim_{r\to\infty} \int_{|z|=r} | u_- |^2 \, ds.$$

The extremal characterization (22) of u_0 is a refinement of the uniqueness theorem for the solution u of (1) and (2) with given boundary values on C. For if $h = 0$ then $|| u_0 || = 0$ and therefore $u_0 \equiv 0$, whence if u satisfies (2) and vanishes on C it must vanish identically, since we can put both $u_+ = u_- = u$ and $u_+ = u_- = iu$ in (21).

A case of special interest for (22) occurs when $h = -e^{ix}$. The function u_+ then represents the scattered field due to a wave of the form e^{ix} incident on C, while u_- represents the scattered field resulting from an incident wave e^{-ix}. The estimate (22) shows that the total scattering cross section σ of C is given by

$$\sigma = \min || u ||^2 \tag{23}$$

among all solutions u of (1) with the boundary values $-e^{ix}$. This characterization of σ indicates clearly the dependence of the scattering cross section on both u_+ and u_-, in view of the form (21) of the extremal function.

Another formula exhibiting the dependence of σ on both u_+ and u_- is obtained when we shift C by an infinitesimal amount δn along its inner normal and attempt to express the scattering cross section σ^* of the shifted curve C^* in terms of quantities associated with C. We continue to take $h = -e^{ix}$ and we introduce the total fields

$$\varphi_+ = e^{ix} + u_+ ,$$
$$\varphi_- = e^{-ix} + u_- ,$$

which vanish on C. We denote by φ_+^* and φ_-^*, and so forth, the corresponding quantities associated with C^*, and we obtain by an easy application of Green's theorem and the radiation condition

$$\begin{aligned}
\sigma^* - \sigma &= \lim_{r\to\infty} \int_{|z|=r} [|u_+^*|^2 - |u_+|^2]\, ds \\
&= \mathscr{I}m \int_{C^*} \bar{u}_+^* \frac{\partial u_+^*}{\partial n}\, ds - \mathscr{I}m \int_C \bar{u}_+ \frac{\partial u_+}{\partial n}\, ds \\
&= \mathscr{I}m \int_{C^*} u_-^* \frac{\partial \varphi_+^*}{\partial n}\, ds - \mathscr{I}m \int_C u_+ \frac{\partial \varphi_-}{\partial n}\, ds \\
&= \mathscr{I}m \int \left(u_-^* \frac{\partial \varphi_+^*}{\partial n} - \varphi_+^* \frac{\partial u_-^*}{\partial n} - u_+ \frac{\partial \varphi_-}{\partial n} + \varphi_- \frac{\partial u_+}{\partial n} \right) ds \\
&= \mathscr{I}m \int \left(u_-^* \frac{\partial e^{ix}}{\partial n} - e^{ix} \frac{\partial u_-^*}{\partial n} - u_+ \frac{\partial e^{-ix}}{\partial n} + e^{-ix} \frac{\partial u_+}{\partial n} \right) ds \\
&= \mathscr{I}m \int \left(\varphi_-^* \frac{\partial \varphi_+}{\partial n} - \varphi_+ \frac{\partial \varphi_-^*}{\partial n} \right) ds.
\end{aligned}$$

In the last integral we are free to choose C as the path of integration, and since $\varphi_+ = 0$ on C and $\varphi_-^* = 0$ on C^*, we find

$$\begin{aligned}
\sigma^* - \sigma &= \mathscr{I}m \left\{ \int_C \varphi_-^* \frac{\partial \varphi_+}{\partial n}\, ds - \int_{C^*} \varphi_-^* \frac{\partial \varphi_+}{\partial n}\, ds \right\} \\
&= -\mathscr{I}m \iint_{D-D^*} \{ \nabla\varphi_-^* \nabla\varphi_+ - \varphi_-^*\varphi_+ \}\, dx\, dy,
\end{aligned}$$

where D^* is the exterior of C^*. We now consider only terms of the first order in the infinitesimal shift δn and derive, with the notation $\sigma^* - \sigma = \delta\sigma$, the Hadamard variational formula

$$\delta\sigma = -\mathscr{I}m \int_C \frac{\partial \varphi_-}{\partial n} \frac{\partial \varphi_+}{\partial n}\, \delta n\, ds \tag{24}$$

for the scattering cross section σ. The result is valid for small shifts δn of either sign.

Formula (24) has an obvious importance for the estimation of the scattering cross section σ of figures near those for which σ has been studied. As a special application of (24), we remark that it substantiates the conjecture that a vertical segment has the least scattering cross section among all simple closed curves which enclose its end-points. In fact, according to (24) we check that by symmetry $\delta\sigma = 0$ for any normal shift of the vertical segment carrying it into a neighboring curve with the same end-points.

Finally, we call attention to the fact that only the most elementary alterations are required in order to extend all the remarks of this section to the case in which the normal derivatives of u are prescribed in the boundary conditions along C. A generalization to space of three dimensions is likewise possible.

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STABILITY OF TWO-DIMENSIONAL PARALLEL FLOWS FOR THREE-DIMENSIONAL DISTURBANCES*

By CHIA-SHUN YIH (*State University of Iowa*)

The object of this note is to establish a relationship between the stability of two-dimensional parallel flows for three-dimensional disturbances and that for two-dimensional ones. The special case of confined flow of a homogeneous fluid has been considered by Squire[1]. In the present note neither is the upper surface of the fluid necessarily assumed to be fixed, nor are the gravitational force and variations in density and viscosity neglected. The variations in density and viscosity, which for two-dimensional flow can occur only in the direction normal to the plane boundary along which the fluid flows, may be continuous or discontinuous.

For the disturbance, a stream function of the type

$$\psi' = \varphi(y) \exp i(mx + nz - mct) \tag{1}$$

can be taken, in which x is measured in the direction of the primary flow, y is measured in the direction normal to the plane boundary, z is taken along an axis normal to the x, y plane, and m, n, and c are constants. If a rotation about the y-axis is performed so that the x'-axis has the direction numbers $(m, 0, n)$ with respect to the original coordinate system, then

$$mx + nz = m'x' \tag{2}$$

in which $m' = (m^2 + n^2)^{1/2}$. If, furthermore, c' is defined by $m'c' = mc$, Eq. (1) can be written as

$$\psi' = \varphi(y) \exp im'(x' - c't) \tag{3}$$

which represents a two-dimensional disturbance progressing in the x'-direction with wave number m' and celerity c'. Similarly, the stream function

$$\psi'' = \varphi(y) \exp i(mx - nz - mct) = \varphi(y) \exp im'(x'' - c't) \tag{4}$$

represents the same disturbance progressing in the x''-direction with direction numbers $(m, 0, -n)$.

Now the stream function

$$\psi = \psi' + \psi'' = 2\varphi(y) \cos (nz) \exp im(x - ct) \tag{5}$$

represents a three-dimensional disturbance progressing in the x-direction with celerity c and having wave numbers m and n in the x- and z-directions, respectively. Since by symmetry ψ' and ψ'' are physically identical, it follows that the flow is stable or unstable for ψ according as it is stable or unstable for ψ', because the differential system governing stability is linear and homogeneous in the quantity representing the stream function of the disturbance, so that the rule of superposition applies.

Now if the Reynolds number, slope, and pressure gradient of the primary flow are

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[1]H. B. Squire, *On the stability for three-dimensional disturbances of viscous fluid flow between parallel walls*, Proc. Roy. Soc. London **A142**, 621-628 (1933).

denoted respectively by R, s, and $\partial p/\partial x$, those for that component of the primary flow pertinent to ψ', denoted by R', s', and $\partial p'/\partial x'$, are determined by

$$m'R' = mR, \qquad m's' = ms, \qquad \frac{\partial p'}{\partial x'} \csc \beta' = \frac{\partial p}{\partial x} \csc \beta \tag{6}$$

in which β and β' are the angles of inclination of the boundary to the horizontal in the x- and x'-directions, respectively, so that $s = \tan \beta$ and $s' = \tan \beta'$. As can be easily seen, the cross flow (in the z'-direction) for the case of ψ' makes no consequential contribution to either the equations of motion or the equation of continuity, and does not affect in any way the satisfaction of the boundary conditions. Hence, the primary flow is stable or unstable for a three-dimensional disturbance according as it is stable or unstable for a two-dimensional one at a lower Reynolds number, a milder slope, and a reduced pressure gradient: the laws of reduction for the three quantities being given by Eqs. (6).

The writer arrived at the foregoing conclusion by using Squire's approach. Whereas many details were obtained as by-products, the rather cumbersome calculations involved are unnecessary if only the principal conclusion is desired. The general approach used here was suggested by the reviewer of the original manuscript of this paper, who attributed it to Professor C. C. Lin of the Massachusetts Institute of Technology.

ON DIFFUSION IN AN EXTERNAL FIELD AND THE ADJOINT SOURCE PROBLEM*

By JULIAN KEILSON (*Lincoln Laboratory, M.I.T.*)

Abstract. If diffusion in an external field is described by $\partial\rho/\partial t = D\nabla^2\rho - \rho/\tau - \nabla\cdot(\mathbf{F}(\mathbf{r})\rho)$, the function $\gamma(r_0)$ describing the probability that a particle at r_0 will reach a collector surface before decaying or being absorbed by other surfaces satisfies the equation $D\nabla^2\gamma - \gamma/\tau + \mathbf{F}(\mathbf{r})\cdot\nabla\gamma = 0$. This equation has no singularity to disturb any geometric symmetry available. Boundary conditions on $\gamma(r)$ at the collector surface and other influencing surfaces are derived and shown to be independent of the external field. The boundary conditions at the secondary surfaces are homogeneous. The collector surface boundary condition is inhomogeneous.

1. Introduction. If particles with lifetime τ diffuse in the presence of an external field $\mathbf{F}(r)$, changes in density $\rho\ (r, t)$ are described by the continuity equation[1]

$$\frac{\partial\rho}{\partial t} = -\frac{\rho}{\tau} - \nabla\cdot\mathbf{j}. \tag{1}$$

Here $\mathbf{j}$, the current density, is given by

$$\mathbf{j} = -D\nabla\rho + \mathbf{F}(\mathbf{r})\rho \tag{2}$$

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[1]J. Keilson, *On the diffusion of decaying particles in a radial electric field*, Journ. Appl. Phys., **24**, 1397–1400 (1953).

so the equation of motion is

$$\frac{\partial \rho}{\partial t} = D\nabla^2 \rho - \frac{\rho}{\tau} - \nabla \cdot (\mathbf{F}(\mathbf{r})\rho). \tag{3}$$

Suppose one is interested in the probability $\gamma(\mathbf{r}_0)$ that a particle at $\mathbf{r}_0$ will reach the collector surface S before decaying or being absorbed by surface S' (Fig. 1).

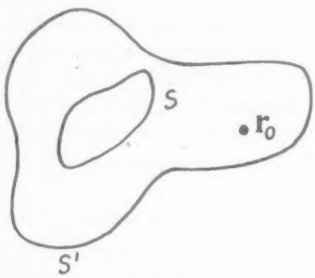

FIG. 1

To obtain $\gamma(r_0)$ one could solve the steady state diffusion equation for a source I_0 at r_0

$$D\nabla^2 \rho - \frac{\rho}{\tau} - \nabla \cdot (\mathbf{F}(\mathbf{r})\rho) = -I_0 \delta(\mathbf{r} - \mathbf{r}_0) \tag{4}$$

with given boundary conditions at surfaces S and S', and obtain $\gamma(r_0)$ as

$$\gamma(r_0) = \frac{1}{I_0} \int_s \{-D\nabla \rho + \mathbf{F}(r)\rho\} \cdot d\sigma. \tag{5}$$

It is often simpler to obtain $\gamma(r_0)$ as the solution of the homogeneous equation adjoint to (4):

$$D\nabla^2 \gamma - \frac{\gamma}{\tau} + \mathbf{F}(\mathbf{r}) \cdot \nabla \gamma = 0. \tag{6}$$

The singularity at $\mathbf{r}_0$ no longer interferes with any geometric symmetry available, and the boundary conditions on γ, it will be shown, no longer contain the field $\mathbf{F}(\mathbf{r})$. The boundary conditions at the collector surface, however, are now inhomogeneous.

That γ obeys Eq. (6) may be seen in the following way. Let $P(\mathbf{r}_0/\mathbf{r}; t)$ be the solution of the time dependent diffusion equation (3), t seconds after the particles were known to be at $\mathbf{r}_0$, i.e., that solution for which

$$P(\mathbf{r}_0/\mathbf{r}; 0) = \delta(\mathbf{r} - \mathbf{r}_0). \tag{7}$$

Continuity of probability demands that, for all t,

$$\gamma(\mathbf{r}_0) = \int P(\mathbf{r}_0/\mathbf{r}; t)\gamma(\mathbf{r})\, d\mathbf{r} + \int_0^t ds \int_S j_n(\mathbf{r}_S, s)\, d\sigma. \tag{8}$$

$j_n(\mathbf{r}_S, s)$ is the normal component of the current density at the point $\mathbf{r}_S$ on our collector surface S at time s. At $t = 0$, the collector current density hasn't had a chance to build up and $j_n(\mathbf{r}_S, 0) = 0$.

If then one differentiates Eq. (8) with respect to t at $t = 0$, it is seen that

$$\int \frac{\partial \rho}{\partial t} (\mathbf{r}_0/\mathbf{r}\,; 0)\gamma(\mathbf{r})\, d\mathbf{r} = 0,$$

i.e.,

$$\int \left\{ D\nabla^2 \delta(\mathbf{r} - \mathbf{r}_0) - \frac{\delta(\mathbf{r} - \mathbf{r}_0)}{\tau} - \nabla\cdot(\mathbf{F}(\mathbf{r})\,\delta(\mathbf{r} - \mathbf{r}_0)) \right\} \gamma(\mathbf{r})\, d\mathbf{r} = 0.$$

When this is integrated by parts, Eq. (6) is obtained.

2. **Boundary conditions.** Suppose the boundary conditions on ρ are

$$j_n = -D \frac{\partial \rho}{\partial n} + F_n(\mathbf{r})\rho = \alpha\rho \text{ at } S, \tag{9}$$

and

$$-D \frac{\partial \rho}{\partial n} + F_n(\mathbf{r})\rho = \alpha'\rho \text{ at } S'. \tag{10}$$

The corresponding boundary conditions on $\gamma(\mathbf{r})$ at S, and S' are wanted. To find them one multiplies (3) by $\gamma(r)$, integrates over space and time, and obtains

$$\int_0^t ds \int \gamma(\mathbf{r}) \frac{\partial P}{\partial t} (\mathbf{r}_0/\mathbf{r}\,; s)\, d\mathbf{r} = \int_0^t ds \int \gamma(\mathbf{r}) \left\{ D\nabla^2 P - \frac{P}{\tau} - \nabla\cdot(\mathbf{F}(\mathbf{r})P) \right\} d\mathbf{r}.$$

If one uses (6) and (7) and carries out integration by parts for both space and time one finds

$$\begin{aligned}\gamma(\mathbf{r}_0) = \int P(\mathbf{r}_0/\mathbf{r}\,; t)\gamma(\mathbf{r})\, d\mathbf{r} &+ \int_0^t ds \int_S \left\{ DP \frac{\partial \gamma}{\partial n} - D\gamma \frac{\partial P}{\partial n} + \gamma F_n(\mathbf{r}_S)P \right\} d\sigma \\ &+ \int_0^t ds \int_{S'} \left\{ DP \frac{\partial \gamma}{\partial n} - D\gamma \frac{\partial P}{\partial n} + \gamma F_n(\mathbf{r}_{S'})P \right\} d\sigma.\end{aligned}$$

But Eq. (8) tells us that

$$\gamma(r_0) = \int P(\mathbf{r}_0/\mathbf{r}\,; t)\gamma(\mathbf{r})\, d\mathbf{r} + \int_0^t ds \int_S \left\{ -D \frac{\partial P}{\partial n} + F_n P \right\} d\sigma.$$

Comparing our last two equations it is seen that one must have for surface S':

$$DP \frac{\partial \gamma}{\partial n} - D\gamma \frac{\partial P}{\partial n} + \gamma F_n P = 0,$$

and hence

$$D \frac{\partial \gamma}{\partial n} + \alpha'\gamma = 0. \tag{11}$$

This means that if, at S', one had perfect reflection ($\alpha' = 0$), the condition on γ there would be $\partial\gamma/\partial n = 0$. If S' is perfectly absorbing ($\alpha' = \infty$), our condition on γ is that $\gamma = 0$. This is what one would expect since a particle near such a surface has little hope of reaching the collector surface.

At the collector surface, to agree with the continuity of probability equation one must have:

$$-D \frac{\partial P}{\partial n} + F_n P = DP \frac{\partial \gamma}{\partial n} - D\gamma \frac{\partial P}{\partial n} + \gamma F_n P$$

and hence

$$\alpha = D \frac{\partial \gamma}{\partial n} + \alpha \gamma. \tag{12}$$

This condition states that at a perfectly absorbent collector surface ($\alpha = \infty$), $\gamma = 1$. If the surface is perfectly reflecting $\partial\gamma/\partial n = 0$, which is consistent with the more natural condition $\gamma = 0$.

A typical application of the formalism above would be to the diffusion of decaying particles in the presence of a radial electric field. In the earlier paper[1] the formalism is applied to the very simple case of a perfectly absorbing sphere about the center of symmetry.

I would like to thank Dr. S. F. Neustadter for his interest and encouragement.

SOME PROPERTIES OF OPTIMAL LINEAR FILTERS[1]

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The purpose of this note is to discuss two considerations that arise in designing a filter that is optimal in the sense of minimizing the integral over time of some norm, under the further condition that we do not wish to employ information about the statistical structure of the signal. This is the "classical" problem of filter design prior to Wiener's work on stationary time series, although, in general, design was and is guided by figures of merit rather than by an explicit minimization procedure. It is probable that the results set forth here are well known, at least intuitively, to designers of servomechanisms, but I have not seen them set forth.

Since the norm to be miniminized is an integral over time, the problem may be viewed as one in the calculus of variations. Suppose that the norm, F, is:

$$F = \int_{t_0}^{t_1} \phi\{y(t), y_{(1)}(t), \cdots, y_{(n)}(t)\}\, dt, \tag{1}$$

where the integrand, ϕ, is a polynomial in some variable, $y(t)$, and its first n derivatives, $y_{(k)}(t) = d^k/dt^k\, y(t)$, $(k = 1, \cdots, n)$; so that we may write:

$$\phi(t) = C + \sum_{i=0}^{n} b_i y_{(i)} + \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij} y_{(i)} y_{(j)} + \psi(t), \tag{2}$$

where $\psi(t)$ is a polynomial composed of terms of third and higher degree in $y(t)$ and its derivatives.

To obtain a necessary condition for a minimum, we form Euler's equation:

$$\phi_0 - \frac{d}{dt}\phi_1 + \cdots + (-1)^n \frac{d^n}{dt^n}\phi_n = 0, \tag{3}$$

where $\phi_k = \partial\phi/\partial y_{(k)}$.

[1]Received April 28, 1954. The results presented in this paper were obtained in the course of research undertaken by the Cowles Commission for Research in Economics under contract Nonr-358(01), NR 047-006 with the office of Naval Research. I am indebted to my colleagues A. Charnes, C. C. Holt, and F. Modigliani with whom I have had numerous valuable discussions on the topic of this note.

Equation (3) is a differential equation of order not greater than $2n$ in $y(t)$. We now make the assumption:
(A). *Equation* (3) *is a linear differential equation with constant coefficients.* From (2), we obtain:

$$\phi_k = b_k + \sum_{i=0}^{n} a_{ik} y_{(i)} + \sum_{j=0}^{n} a_{kj} y_{(j)} + \psi_k(t) \qquad (k = 0, \cdots, n), \tag{4}$$

$$\frac{d^k}{dt^k}\phi_k = \sum_{i=0}^{n} a_{ik} y_{(i+k)} + \sum_{j=0}^{n} a_{kj} y_{(j+k)} + \frac{d^k}{dt^k}\psi_k(t) \qquad (k = 1, \cdots, n). \tag{5}$$

Substituting (4) and (5) in (3), we have, under assumption (A):

$$\begin{aligned} b_0 &+ \sum_{i=0}^{n} a_{i0} y_{(i)} + \sum_{j=0}^{n} a_{0j} y_{(j)} + \psi_0(t) \\ &+ \sum_{k=1}^{n} (-1)^n \left\{ \sum_{i=0}^{m} a_{ik} y_{(i+k)} + \sum_{j=0}^{n} a_{kj} y_{(j+k)} + \frac{d^k}{dt^k}\psi_k(t) \right\} \\ &= \alpha_0 y + \alpha_1 y_{(1)} + \cdots + \alpha_{2n} y_{(2n)} = 0. \end{aligned} \tag{6}$$

Rearranging the terms of (6), we obtain:

$$\begin{aligned} b_0 &+ \sum_{k=0}^{n} (-1)^k \left\{ \sum_{i=0}^{n} (a_{ik} y_{(i+k)} + a_{ki} y_{(i+k)}) \right\} \\ &+ \psi_0(t) + \sum_{k=1}^{n} (-1)^n \frac{d^k}{dt^k}\psi_k(t) - \sum_{g=0}^{2n} \alpha_g y_{(g)} = 0. \end{aligned} \tag{7}$$

Now, since all the terms of ψ are of third or higher degree in y and its derivatives, all the non-vanishing terms of ψ_k will be of second or higher degree, as will also all the terms of $d^k\psi_k/dt^k$. Hence, the linearity of the differential equation implies that:

$$\psi_0(t) + \sum_{k=1}^{n} (-1)^n \frac{d^k}{dt^k}\psi_k(t) = 0. \tag{8}$$

A necessary and sufficient condition for (8) [see Courant-Hilbert, vol. I, p. 167] is that:

$$\int_{t_0}^{t} \psi(y, \cdots, y_{(n)})\, dt = \xi(y, \cdots, y_{(n-1)}), \tag{9}$$

i.e., that ψ is independent of the path and hence does not affect the optimum.

We have proved the

Theorem I: A necessary and sufficient condition that the extremal corresponding to (2) satisfies assumption (A) is that:

$$F = \int_{t_0}^{t} \left[c + \sum_{i=0}^{n} b_i y_{(i)} + \sum_{i=0}^{n} \sum_{j=0}^{n} a_{ij} y_{(i)} y_{(j)} \right] dt + \xi(y, \cdots, y_{(n-1)}). \tag{10}$$

From (7) we can deduce:

Theorem II: In $\sum_{g=0}^{2n} \alpha_g y_{(g)}$, all α_g must vanish for odd g.

Proof: Since all summations in (7) run from zero to n, for each term $(-1)^k\ (a_{ik} + a_{ki})\ y_{(i+k)}$ we have a corresponding term $(-1)^i(a_{ki} + a_{ik})\ y_{(k+i)}$. If $(i + k)$ is an odd number these terms will have opposite sign and, being otherwise identical, will vanish.

Theorem II has an important consequence. Form the characteristic equation for (7):

$$\sum_{k=0}^{n} \alpha_{2k} p^{2k} = 0. \tag{11}$$

Because this is an even function, if p_1 is a root, $-p_1$ is also a root. Hence, either (1) all the roots are pure imaginaries, or (2) there is at least one root with positive real part. If we interpret the system described by the Euler equation as a linear "filter" or decision rule, the behavior produced by application of the rule will be dynamically unstable.

The two theorems we have established may be interpreted as follows:

1. From Theorem I it appears that linear filters will have desirable qualities when the norm or "error" we wish to minimize is quadratic. If the norm is decidedly non-quadratic this suggests that we can improve filter performance by introducing appropriate non-linearities.

2. From Theorem II and the consequence derived from it, it appears that straightforward application of the calculus of variations to the filter design problem leads to the prescription of an unstable filter, and hence is not practicable.

It can easily be shown that point (2) is related to the fact that the Euler equations give only a necessary and not a sufficient condition for a minimum. Hence it does not follow that a path, $y(t)$, that satisfies (3) will thereby minimize (1). The specification of appropriate initial and terminal conditions to guarantee a *bona fide* minimum requires information about the future of the signal, and therefore, in the face of an incompletely predictable signal the method breaks down.

The difficulty cannot be avoided by minimizing in the domain of the Laplace transform of ϕ instead of in the time domain. For, in general, by Parseval's theorem:[2]

$$\int_{-\infty}^{\infty} [\phi(t)]^2 \, dt = \int_{-\infty}^{\infty} [\phi^*(p)]^2 \, dp, \tag{12}$$

where $\phi^*(p)$ is the transform of $\phi(t)$.

Hence, any function that is a stationary value for the right-hand side of (12) is simply the transform of the function that is a stationary value for the left-hand side.

It is beyond the scope of the present note to discuss newer methods that avoid this difficulty without reverting to cut-and-try procedures or the use of arbitrary figures of merit.

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[2]Professor A. Charnes pointed out to me the relevance of Parseval's theorem for this problem. Minimization in the domain of the transform is essentially the method of Ritz for the "direct" solution of variational problems (Courant-Hilbert, I: p. 150).

BOOK REVIEWS

History of strength of materials with a brief account of the history of theory of elasticity and theory of structures. By S. P. Timoshenko. McGraw-Hill Book Co., Inc., New York, Toronto, London. 1953. x + 452 pp. \$10.00.

Professor Timoshenko's original contributions to the subject of Strength of Materials are outstanding in contemporary literature; the same is true of his various books on Theoretical Mechanics. It is therefore

with considerable interest that a new book by him is read by persons sharing his interests. This is perhaps even more true of the present book. The author has delayed writing it until he had the opportunity to devote an uninterrupted period of time in uncovering all the necessary historical material.

The book is founded upon lectures on the History of Strength of Materials which the author has given over the last twenty-five years to engineering students with some knowledge of the strength of materials and theory of structures. In writing this book the author has had in mind not only such students but also other people whose immediate interests are not specifically in this field. The histories of the three subjects, strength of materials, theory of elasticity, and theory of structures, are so closely interwoven as to be almost inseparable from each other. Much has been written in this historical field from various points of view. The author has followed the example of Saint-Venant in his "Historique Abrégé . . ." rather than that of Todhunter and Pearson in their "A History of the Elasticity and Strength of Materials". The result is an informal account of the history of strength of materials. Included are closely-related parts of the histories of the theory of elasticity and the theory of structures, purely mathematical developments being omitted from the former and purely technical developments from the latter. The presentation, with few deviations, is chronological. Seven chapters are devoted to the strength of materials, two to the theory of structures, and four to the theory of elasticity. These last chapters omit the mention of some well-known names, such as Volterra, and should not be read separately.

A brief introduction describes the empirical approach to the strength of materials current from Ancient Times right down to the Renaissance, and includes comments on the failure of early Renaissance engineers to appreciate da Vinci's work. The story proper begins with Galileo and unfolds through many equally famous names right down to the present time. The wealth of material may be judged from these statistics. About 675 names are mentioned in the text, and biographical material is given for nearly 80 of these. There are 245 figures, and about 40 of these are reproductions of likenesses of famous men. The reader can hardly fail to be stimulated by the biographical accounts often enlivened by anecdotes. The presentation inevitably presents a more difficult task as knowledge accumulates through the efforts of an ever-increasing body of research workers. This is especially true of the period 1900-50 dealt with in the final three chapters. This period is of course the one with which the author has been intimately concerned, and the developments are ably described. Here the coverage is extremely wide; thus fatigue of metals, approximate methods of solving elasticity problems, and the theory of ship structures are some examples of topics discussed. Many original references are cited throughout the book, and discussion is often given of the influence of great teachers. Comments are also made on the circumstances that either stimulated or retarded progress in the strength of materials in different countries over similar periods of time.

A stranger to the subject would not suspect, although outside the covers of this book it is well-recognized, that the genesis of the modern development of the strength of materials in the United States is largely attributable to Professor Timoshenko himself. In the text the author simply refers briefly to himself as a pupil of Prandtl; his name does not occur in the index, and is relegated exclusively to footnote references to original papers.

The reviewer agrees with the author that there is a place for formal lectures on such historical developments in any well-balanced engineering curriculum. For pedagogic reasons it is often not possible to follow the chronological development of a subject in a course of lectures to students. Engineering students must therefore often fail to appreciate the painstaking way in which our present knowledge of the strength of materials has been gradually built-up since Galileo's time. The current plea is a reflection of the need for the increased teaching of the history and philosophy of the natural sciences in undergraduate curricula. The reviewer regrets the author's strict preference for the presentation of the research worker's results in a manner directly of use to the engineer.

The reviewer has detected but few misprints and errors in statements, although some must be almost inevitable in a work of this kind. Kelvin's college is stated to be St. Peter's; the present name is Peterhouse.

H. G. Hopkins

Anniversary volume on applied mechanics dedicated to C. B. Biezeno by some of his friends and former students on the occasion of his sixty-fifth birthday, March 2, 1953. N. V. De Technische Uitgeverij H. Stam, Haarlem, 1953. 328 pp. and 2 fold-in plates. $5.60.

The beautifully made-up volume contains a brief biography and a list of publications of C. B. Biezeno and the following articles: W. Boomstra, Triangles équilatères inscrits dans une conique donnée; H. Bremekamp, Sur la théorie de Sturm-Liouville; Th. v. Kármán and G. Millan, The thermal theory of constant pressure deflagration; J. M. Burgers, Some remarks on detonation and deflagration problems in gases; C. Koning, Some interference problems; R. V. Southwell and Gillian Vaisey, A problem suggested by Saint-Venant's Mémoire sur la torsion des prismes; R. Grammel, Nichtlineare Schwingungen mit unendlich vielen Freiheitsgraden; R. J. Legger, The d'Alembert principle; A. D. de Pater, La stabilité d'un dicône se déplacant sur une voie en alignement droit; A. van der Neut, The local instability of compression members built up from flat plates; J. A. Haringx, Stresses in corrugated diaphragms; A. van Wijngaarden, Ut tensio sic vis; D. Dresden, Shrink-fit used to transmit a torque; W. T. Koiter, On partially plastic thick-walled tubes; G. G. J. Vreedenburgh and O. Stokman, Some new elements in the calculation of flat slab floors; J. P. Mazure, Statical problems in the code of practice for steel windows; J. J. Koch, The Laboratory for Applied Mechanics at the Technological University of Delft; R. G. Boiten, The design of diaphragms for pressure measuring devices based on the use of wire-electrical straingauges.

W. Prager

The dynamics and thermodynamics of compressible fluid flow. Volume I. By Ascher H. Shapiro. The Ronald Press Co., New York, 1953. xiii + 647 pp. $16.00.

Professor Ascher H. Shapiro of the Massachusetts Institute of Technology has undertaken a very important and useful task in writing a treatise on compressible fluids. As the title suggests, it is impossible in such a study to separate the dynamical from the thermodynamical point of view and this accounts for both the difficulty and the interest of this branch of mechanics of continua. The subject dealt with in this book is of great practical importance. At the present time, aeronautical engineers, mechanical engineers, chemical engineers, applied physicists and applied mathematicians frequently need to investigate and to apply the theory of compressible fluids. They will find in this book material of much interest. The author, a professor of mechanical engineering, has given a number of practical applications to various branches of engineering science in each chapter. He has planned this treatise in two volumes, the whole divided into 8 parts and 28 chapters. So far only the first volume containing the first four parts and part of the fifth has been published. The size of the volume (647 pp.) gives some indication of the amount of information contained in it.

Part I sets forth the basic concepts and principles from which the remainder of the book proceeds. The first chapter—foundations of fluid dynamics—contains definitions of a fluid and of the continuum properties (density, pressure, viscosity) and discussions of the conservation of mass and momentum theorems applied to a control volume. In the second chapter—foundations of thermodynamics—the author reviews the most important definitions and concepts of thermodynamics and states the two laws. The second law is introduced by means of the definition of the thermal efficiency of heat engines and is related to the impossibility of obtaining positive work from a system passing through a complete cycle while exchanging heat with only a single source. The thermodynamic properties of the continuum are then defined (internal energy, enthalpy, etc.) and the theorem of conservation of energy is obtained by application of the first law to a control volume. The chapter ends with the basic formulae for a perfect gas. In the third chapter which concludes Part I are introduced the principal ideas concerning compressible fluids. The speed of sound is defined by its physical significance (velocity of propagation of a plane pressure pulse) and the physical differences between subsonic and supersonic flows are then emphasized by statement of the three Kármán rules for supersonic flow and are discussed with the help of various schlieren and interferometer photographs. This chapter also includes a study of the similarity parameters and a survey of the optical methods which were used in the experiments (interferometer, shadowgraph and schlieren methods).

Part II is devoted to "one-dimensional flow"; although it is restricted to the steady case (the unsteady case will be treated in Volume II) it is almost 200 pages long. In the first chapter of this part the fluid is assumed to be continuous and isentropic, and a complete discussion for converging or converging-diverging nozzles and supersonic diffusers is given, including a special study of the choking effect. In the next chapter, the theory of normal shock waves and the basic formulae relating to this theory are found. Special attention is given to the physical explanation of the formation of shock waves, and some discussion is given of the thickness of these shocks. The ideas developed in this chapter are applied to nozzles, supersonic diffusers and the supersonic Pitot tube. Then a chapter is devoted to the detailed study of flow in constant-area ducts with friction. It is shown how friction can produce choking even though the duct area remains constant. Information on isothermal flow in long ducts is also given. The following chapter is devoted to flow in ducts with heating or cooling, i.e., to the case in which, though the area is kept constant and the friction is neglected, the stagnation enthalpy may be changed for various reasons. There exist many problems in engineering in which such a situation arises. Finally, in the last chapter of Part II, more general problems are discussed in which two or more of the previously mentioned phenomena may appear.

Part III contains an introduction to flow in two and three dimensions. It gives the basic definitions and theorems: circulation, rotation and their physical significance; Euler's equation; Kelvin's and Bernoulli's theorems; continuity equation; equation of velocity potential; etc.

Part IV deals with subsonic flow. The author begins with the so-called linearized theory in which small perturbations are superposed on a uniform flow. After the classical example of flow past a wave-shaped wall, the fundamental similarity rules are given. These allow us to relate the subsonic compressible flow past a certain profile to the incompressible flow past a second profile derived from the first by means of an affine transformation. The famous Prandtl-Glauert rule is discussed and many applications are made, in particular to wind tunnel corrections. The next chapter is devoted to the hodograph method for two-dimensional subsonic flow, which in fact allows us to linearize the problem without introducing new simplifications. The tangent-gas approximation of Chaplygin is discussed and applications lead to the Kármán pressure correction formula. In the following chapter miscellaneous methods and results are given—method of expansion in series in terms of the Mach number (Rayleigh-Janzen), method of expansion in series in terms of a shape parameter, relaxation method, etc. Comparisons of theoretical results with experiments are made. The last chapter of Part IV is devoted to three-dimensional subsonic flow—flow past spheres, ellipsoids, bodies of revolution, wings of finite span and sweptback wings. The extension of the similarity rules is made with special care.

Part V deals with two-dimensional supersonic flow (only the first three chapters of this part are included in Volume I). The linearized theory is treated in great detail in order to facilitate its application to practical problems and to provide a good introduction to the following chapter which is devoted to the method of characteristics. This last method is presented as a natural generalization of the basic results of linearized theory. Application of the method is made to the jet problem and to the design of a supersonic wind tunnel; the case in which rotation is present is also considered. In the last chapter, the theory of oblique shock waves is developed (shock equations, reflection and interaction of shocks). Many examples of flow containing shocks are found, together with experimental results. The book ends with an appendix devoted to a mathematical formulation of the theory of characteristics and with some useful tables (standard atmosphere, isentropic flow, normal shock, one-dimensional flow with friction or with change in stagnation enthalpy, hodograph characteristic functions for supersonic flow).

Each chapter in the book begins with a clear introduction advising the reader of the aim of the study; it is followed by a nomenclature list explaining the significance of the symbols used in the chapter. After each article, "working formulae and charts" are given which can be consulted when a specific application is planned. At the end of each chapter there are several well selected problems most of which are drawn from practical situations arising in engineering science. A selected bibliography is appended to each chapter.

On the whole, the material contained in this book is well selected and clearly presented. The second part which is devoted to one-dimensional flow is particularly well developed. Because the book is written primarily for engineers, the viewpoint of physics and engineering predominates, emphasis being placed on results rather than on proofs. For the most part, the author attempts to make each chapter a self-contained unit with the result that some formulae or theorems are proved more than once. The order is not always the most logical. For example, the equation of continuity for two- and three-dimensional flows is given on p. 283—after Euler's equation, Bernoulli's theorem, Kelvin's theorem and Crocco's theorem;

the reviewer suspects that this equation may have been used implicitly before its formal presentation. Moreover, in discussing the linearized theory, the linearization of the boundary conditions does not appear to be carried out systematically. Also, it is not immediately clear why the author claims that the first of the three similarity laws for two-dimensional subsonic flow given on p. 323 is the most exact. However, only a few such criticisms can be made.

The general level of the book is perhaps too high for it to be of value as a classical textbook but those who teach the subject of compressible fluid flow will undoubtedly find it immensely useful. Some years ago, Courant and Friedrich's classical book provided a mathematical approach to the theory of compressible fluids; this new book provides a more physical and technical approach to the same subject. Together, these two extensive treatises constitute a firm basis for the study of this subject.

P. Germain

Methods of theoretical physics. By Philip M. Morse and Herman Feshbach. Volumes I and II. McGraw-Hill Book Company, Inc., New York, Toronto, London, 1953. xxii + 997 pp. (Vol. I), xviii + 980 pp. (Vol. II). $30.00 set.

In more than 1900 pages and two volumes the authors have packed a formidable amount of material on the subject of mathematical methods in physics. This treatise is largely concerned with fields and boundary value problems of all types. The discussion of techniques in connection with field problems naturally leads to the newer aspects of perturbation theory and variational methods; the reader's attention is called particularly to Chapters 9 and 11 in this connection (see topics listed below). The section on Green's functions is really excellent as are many other sections of this work. Anyone with strong interest in the use of advanced mathematical methods will find this work extremely valuable. One of its advantages is that it is rather complete and self-contained.

The advantages mentioned become disadvantages when one considers using this treatise as a text. On the other hand the work was the outgrowth of a course given for quite a few years by the authors at M. I. T.

The chapter headings are listed below together with section headings within each chapter and the space devoted to each chapter. Chapter 1 (117 pages) Types of Fields: Scalar Fields, Vector Fields, Curvilinear Coordinates, The Differential Operator Del, Vector and Tensor Formalism, Dyadics and Other Vector Operators, The Lorentz Transformation, Four-Vectors, Spinors. Chapter 2 (155 pages) Equations Governing Fields: The Flexible String, Waves in an Elastic Medium, Motion of Fluids, Diffusion and Other Percolative Fluid Motion, The Electromagnetic Field, Quantum Mechanics. Chapter 3 (73 pages) Fields and the Variational Principles: The Variational Integral and Euler Equations, Hamilton's Principle and Classical Dynamics, Scalar Fields, Vector Fields. Chapter 4 (133 pages) Functions of a Complex Variable: Complex Numbers and Variables, Analytic Functions, Derivatives of Analytic Functions, Multivalued Functions, Calculus of Residues, Asymptotic Series: Method of Steepest Descent, Conformal Mapping, Fourier Integrals. Chapter 5 (164 pages) Ordinary Differential Equations: Separable Coordinates, General Properties and Series Solutions, Integral Representations, Table of Separable Coordinates in Three Dimensions, Second-Order Differential Equations and Their Solutions. Chapter 6 (115 pages) Boundary Conditions and Eigenfunctions: Types of Equations and of Boundary Conditions, Difference Equations and Boundary Conditions, Eigenfunctions and Their Use, Table of Useful Eigenfunctions and Their Properties, Eigenfunctions by the Factorization Method. Chapter 7 (100 pages) Green's Functions: Source Points and Boundary Points, Green's Functions for Steady Waves, Green's Function for the Scalar Wave Equation, Green's Function for Diffusion, Green's Function in Abstract Vector Form, Table of Green's Functions. Chapter 8 (100 pages) Integral Equations: Integral Equations of Physics, General Properties of Integral Equations, Solution of Fredholm Equations of the First Kind, Solution of Integral Equations of the Second Kind, Fourier Transforms and Integral Equations, Tables of Integral Equations and Their Solutions.

Volume II. Chapter 9 (172 pages) Approximate Methods: Perturbation Methods, Boundary Perturbations, Perturbation Methods for Scattering and Diffraction, Variational Methods, Tabulation of Approximate Methods. Chapter 10 (158 pages) Solutions of Laplace's and Poisson's Equations: Solutions in Two Dimensions, Complex Variables and the Two-Dimensional Laplace Equation, Solutions for Three Dimensions, Trigonometric and Hyperbolic Functions, Bessel Functions, Legendre Functions.

Chapter 11 (252 pages) The Wave Equation: Wave Motion on One Space Dimension, Waves in Two Dimensions, Waves in Three Space Dimensions, Integral and Variational Techniques, Cylindrical Bessel Functions, Weber Functions, Mathieu Functions, Spherical Bessel Functions, Spherical Bessel Functions, Spheroidal Functions, Short Table of Laplace Transforms. Chapter 12 (173 pages) Diffusion, Wave Mechanics: Solutions of the Diffusion Equation, Distribution Functions for Diffusion Problems Solutions of Schroedinger's Equation, Jacobi Polynomials, Semi-cylindrical Functions. Chapter 13 (143 pages) Vector Fields: Vector Boundary Conditions, Eigenfunctions and Green's Functions, Static and Steady-state Solutions, Vector Wave Solutions, Table of Spherical Vector Harmonics.

With each chapter a set of problems is offered.

The seventeen tables at the end of Volume II have the following headings: Trigonometric and Hyperbolic Functions; Hyperbolic Tangent of Complex Quantity; Inverse Hyperbolic Tangent of Complex Quantity; Logarithmic and Inverse Hyperbolic Functions; Spherical Harmonic Functions; Legendre Functions for Large Arguments; Legendre Functions for Imaginary Arguments; Legendre Functions of Half-integral Degree; Bessel Functions for Cylindrical Coordinates; Hyperbolic Bessel Functions; Bessel Functions for Spherical Coordinates; Legendre Functions for Spherical Coordinates; Amplitudes and Phases for Cylindrical Bessel Functions; Amplitudes and Phases for Spherical Bessel Functions; Periodic Mathieu Functions; Normalizing Constants for Periodic Mathieu Functions and Limiting Values of Radial Mathieu Functions.

ROHN TRUELL

The aim and structure of physical theory. By Pierre Duhem. Translated from the French by Philip P. Wiener. Princeton University Press, Princeton, N. J., 1954. xxii + 344 pp. $6.00.

This book is a translation of the original work of the author entitled *La Théorie Physique: Son Object, Sa Structure* (Marcel Rivière et Cie, Paris, 1914) and is the first of Duhem's writings on the philosophy of science to appear in the English language. Actually the French version from which the translation has been made is the second edition of a still earlier work, published in 1906.

Pierre Duhem (1861-1916) was a well known French thermodynamical physicist of the late nineteenth century and made notable contributions to the application of thermodynamics to physical chemistry in developing the ideas of Gibbs, which indeed he was among the first to popularize in France. At the same time he took a profound interest in the history and philosophy of physical science. Though somewhat overshadowed in this field by his more brilliant and versatile contemporary, Henri Poincaré, he nevertheless established a well justified reputation as an incisive critic of physical methodology along with men like Mach, Helmholtz, Clifford and Pearson.

The methodology of physics has been the subject of serious investigation during the past half century and to a certain extent this inevitably serves to date the treatise of Duhem. Nevertheless, it appears he still has something helpful to say to his successors and his essentially pragmatic, positivistic viewpoint of the deductive character of physical theory will prove very stimulating to the present day reader.

The author's treatment is divided into the two main sections reflected in the title. In the first he discusses the relation between physical theory and metaphysics, physics and the "natural" classificatio n of experience, the history of the development of theories, and the relation between abstract theories and mechanical models. The second part is devoted to an analysis of the structure of a physical theory from the standpoints of the notions of quantity and quality, the use of mathematics, the relation of theory to experiment, and the choice of hypotheses. There are two appendices, the first of which on "The Physics of a Believer", though not at all a detailed discussion of the relation between physical science and religion presents some interesting side lights on this perennial question.

The clarity and vividness of Duhem's style have been very ably caught and preserved by the translator, and the thoughtful reader will never mistake the author's meaning. Though in general rather judicious in his estimates, he was sufficiently strong-minded to develop a few obsessions which give a personal flavor to his work. For example, Duhem firmly believed that English physicists approach the building of physical theories in quite different fashion from their Continental colleagues. His arguments are persuasive but will scarcely stand up in the light of more recent history. Moreover his fervent optimism that physical theory progresses by steady evolution to an ideal end, the "natural classification"

of experience, apparently failed to foresee the unparalleled creation of new experience in modern physical laboratories which is giving the present day theorist nightmares. Of course we can all agree with Duhem that physical theories will continue to be constructed and that some of them will be successful, but it seems undesirable to hamper ourselves by any preconceived metaphysical notion of an "ideal end".

This book can be recommended in unqualified terms to all who are interested in the logical structure of physical science.

R. B. Lindsay

Applied elasticity. By Chi-Teh Wang. McGraw-Hill Book Co., New York, Toronto, London, 1953. ix + 357 pp. $8.00.

This elasticity text is clearly intended for the use of the advanced engineering student, as is stated in the preface. The classical theory of elasticity is well presented in a manner that presupposes no extensive knowledge of advanced calculus. Where necessary, an attempt is made to introduce whatever higher mathematics is needed. For the most part, this attempt is successful so far as both rigor and technique of solution is concerned, keeping in mind that the main purpose is the latter. However, the introduction to the theory of complex variables is, I believe, much too hurried and sketchy; and at some points certainly does not possess the rigor the author hopes is not sacrificed by concentration on methods of solution.

Chapters I and II develop the concepts of stress and strain, respectively, in straightforward fashion. Chapter III is concerned with stress-strain relations and the concept of strain energy. Chapter IV sets up the plane-stress and plain-strain problem in terms of Airy's stress function and solves problems by choosing solutions to the biharmonic equation. Rotating disks and thermal stresses are also discussed. Chapter V treats the torsion problem in terms of the warping function. The stress function is introduced to make use of the membrane analogy in the solution of the torsion problem of thin open sections and thin tubes.

Since one of the laudable purposes of the book is to provide techniques of solution where exact analytic solutions are intractable, much space is devoted to numerical methods. Chapter VI discusses finite-difference approximations and relaxation methods, with application to the torsion problem as an example. In Chapter VII, energy principles and variational methods are discussed analytically. The results of Chapters VI and VII are used to obtain numerical solutions of buckling problems in Chapter X, the analytic formulation and solution of which are discussed in Chapter IX.

Chapter VIII uses complex variables to obtain solutions to torsion problems and plane problems.

Chapters XI and XII discuss bending and buckling of thin plates, and the theory of thin shells, respectively. The latter is preceded by a comparatively extensive discussion of the differential geometery of a surface.

The book is clearly written, easy to read, and on the whole does, I believe, fulfill the author's purpose "... to provide the student with the necessary fundamental knowledge of the theory ... to acquaint him with the most useful analytical and numerical methods (of solution)."

Harry J. Weiss

Stability theory of differential equations. By Richard Bellman. McGraw-Hill Book Co., New York, Toronto, London, 1953. xiii + 166 pp. $5.50.

This attractive little book gives an original and useful survey of various aspects of behavior, as $t \to \infty$, of solutions of systems of ordinary differential equations. After an introductory chapter introducing the convenient vector-matrix-norm notation for treating such systems, the author defines the sense in which he uses the word stability, as follows:

"*Definition*. The solutions of

$$\frac{dy}{dt} = A(t)y \tag{7}$$

are stable with respect to a property P and perturbations $B(t)$ of type T if the solution of

$$\frac{dz}{dt} = (A(t) + B(t))z \tag{8}$$

also possess property P. If this is not true, the solutions of (7) are said to be unstable with respect to property P under perturbations of type T."

This definition can be applied to many problems, concerning non-linear (when rephrased) as well as linear differential equations. In many cases (e.g., if $A(t)$ is constant), the pattern followed by the author consists in showing that the qualitative asymptotic behavior of known special solutions is unaffected by "small" perturbations of the coefficients—Liapounoff's famous stability theorem is a special case. In other cases, striking counterexamples to plausible guesses are given. In still others, ingenious isolated results are "rescued from oblivion" by displaying them in easily accessible form (e.g., the Fowler-Emden equation, to which Chap. VII is devoted).

The reader who is looking for results of the type just described should first consult Bellman's book. If he does not find them there, he will probably locate them by consulting the well-organized bibliography.

On the other hand, there are a number of important topics, whose omission is not suggested by the somewhat misleading title of this book. For example, the basic Routh-Hurwitz stability criteria are nowhere mentioned, being replaced throughout by the phrase "Let all solutions of $dy/dt = Ay$ tend to zero as $t \to \infty$." Again, no discussion is given of topological arguments or of the stability of limit cycles in the Rayleigh-van der Pol equation. In treating the simple second-order equation $(ku')' + ru = 0$, deeper methods (saddle-point method, asymptotic expansions in λ, etc.) are not explained.

Perhaps because of his exclusive concern with a single problem and with formal methods, the author achieves an admirable clarity and uniformity of style. The result is a very useful survey of the qualitative asymptotic theory of ordinary differential equations.

GARRETT BIRKHOFF

Handbook of elliptic integrals for engineers and physicists. By P. F. Byrd and M. T. Friedman. Springer Verlag, Berlin-Gottingen-Heidelberg, 1954. xii + 355 pp. \$8.58 (paperbound), \$9.44 (clothbound).

This useful handbook contains over 3000 integrals and formulas and 28 pages of numerical tables designed to aid engineers and physicists in the evaluation of elliptic integrals that occur in practical problems. The explanatory material is written on an elementary level and does not require previous acquaintance with elliptic integrals or functions. The notations of Legendre and Jacobi are used in preference to that of Weierstrass. The typographical presentation, of particular importance in a work of this character, is excellent. A considerable saving of space has been achieved by the use of 9-point type in numerators and denominators of fractions. The resulting compactness of formulas greatly facilitaties the use of the handbook.

W. PRAGER

Proceedings of the Eastern Joint Computer Conference. The Institute of Radio Engineers, Inc., New York, 1954. 125 pp. \$3.00.

This report contains the Papers and Discussions presented at the Joint I.R.E.-A.I.E.E.-A.C.M. Computer Conference held in Washington, D. C., in December 1953. The Conference was the third of its kind and the theme chosen for it was, "Information Processing Systems—Reliability and Require-

ments." The papers can be classified in three groups: (1) Statements of data processing and calculation requirements. Descriptions of requirements for the Life Insurance Business, Numerical Weather Prediction, Air Traffic Control and Linear Real-Time Systems are given. (2) Papers describing the operational and performance characteristics of existing machines. Operating experiences with the OARAC, the Los Alamos 701, a large REAC installation, the UNIVAC, the SEAC and the ORDVAC are described. Two papers deal with performance tests, the "National Bureau of Standards Performance tests" and the "Acceptance Tests for the Raythorn Hurricane Computer." Two further papers can be placed in this group. They are descriptions of the MIT Magnetic-Core Memory and the Electrostatic Memory of ILLIAC. (3) The third group of papers is descriptive of work carried out in connection with the reliability of components; Magnetic Tape, Electrolytic Capacitors, Resistors, Electron Tubes, and other electronic units are discussed.

Two very interesting papers, one by H. H. Goldstine and the other by J. W. Mauchly, do not conveniently fall into one of the above catagories. Goldstine gives a brief resumé of von Neumann's work on logical design and Mauchly discusses the advantages of built-in checking.

The report as a whole is well printed and readable.

J. Foulkes

Linear operators spectral theory and some other applications. By Richard G. Cooke. MacMillan and Co., Ltd., London, 1953. xii + 454 pp. $10.00.

The topics presented in this book range over the spectral theory of operators in Hilbert space, an introduction to quantum mechanics, a valuable discussion of Banach algebras and certain ideas on sequence spaces.

The Hilbert space theory is introduced by a discussion of $\mathfrak{H}_2$ and of abstract Hilbert space. The usual elementary operator theory is covered and the spectral theory due to von Neumann using the Cayley transform. The Stone theory of spectra and resolvent, a development of the Hallinger theory due to Lengyel, the Cooper theory based on integrating the differential equation for the Stone group, and the Riesz-Lorch proof are given.

The presentation tends to be non-geometrical, with a preference for analytical and matrix methods. This permits a simple and direct connection with quantum mechanics. The harmonic oscillator is treated both in the Heisenberg and Schrödinger formulation and perturbation theory is given also in matrix terminology.

The analytic and non-geometric viewpoint is further developed in Chapter 6, which is entitled "Projective convergence and limit in matrix spaces and rings." The basis of this chapter is certain definitions of limits for sequences, which are given in another book of the author, i.e., R. G. Cooke, "Infinite matrices and sequence spaces," MacMillan and Co., (1950), and not repeated in the present work. A number of topologies for sequence spaces are introduced and these in turn lead to topologies for the matrices associated with linear transformations between sequence spaces. The work of H. S. Allen is compared with that of Köthe and Toeplitz and Banach's results, specialized to sequence spaces. Greater generality is claimed for Allen's notions, but they apply, of course, only to the sequential situation. However, the discussion of Chapter 6 seems to the reviewer to be of a far more specialized character than that of the other chapters and of less general interest.

Chapter 7 is on Banach algebras and is quite fascinating. The initial sections contain Zron's Lemma, the Hahn Banach Theorem, a discussion of weak convergence of linear functionals and an algebraic discussion of maximal ideals. The objective is the development of the Gelfand theory in the case of commutative Banach algebras. This theory is applied to yield certain results of Wiener's on the convergence of the Fourier series and the Fourier integral for a reciprocal of a function. It is also used as a tool in the investigations of topological spaces by means of the set of continuous functions defined on them. This chapter also contains a discussion of Radon measure and a proof of Wiener's theorem on the density of the translations of a function in L.

It should be clear from the above that this represents on the whole an interesting and useful collection of topics in the mathematical theory of operators. Most graduate students should be able to read the book without difficulty. A useful set of references for further reading is also given.

F. J. Murray

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THIRD U. S. NATIONAL CONGRESS OF APPLIED MECHANICS

Preliminary Announcement

The Third U. S. National Congress of Applied Mechanics will be held at Brown University, Providence, Rhode Island, during June 11-14, 1958. It is hoped that the scheduling of conflicting meetings can be avoided by this early announcement of the date chosen for the Congress. Further announcements concerning the preparation of papers will be made as the Congress draws nearer.

Inquiries regarding the Congress should be addressed to one of the following members of the Organizing Committee at Brown University, Providence 12, R. I.: Professor D. C. Drucker, Secretary; Professor E. H. Lee, Treasurer; Professor W. Prager, Chairman.

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